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Oscillatory radial solutions for subcritical biharmonic equations

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ABSTRACT

It is well known that the biharmonic equation $\Delta^2 u = u|u|^{p-1}$ with $p \in (1, \infty)$ has positive solutions on $\mathbb{R}^n$ if and only if the growth of the nonlinearity is critical or supercritical. We close a gap in the existing literature by proving the existence and uniqueness, up to scaling and symmetry, of oscillatory radial solutions on $\mathbb{R}^n$ in the subcritical case. Analyzing the nodal properties of these solutions, we also obtain precise information about sign-changing large radial solutions and radial solutions of the Dirichlet problem on a ball.

1. Introduction

The equation $-\Delta u = u|u|^{p-1}$ and, to a lesser degree, $\Delta u = u|u|^{p-1}$, with $p \in (1, \infty)$ and Δ the n-dimensional Laplacian (for some $n \in \mathbb{N}$), are among the most extensively studied second-order semilinear elliptic equations, serving as paradigms for a wide class of problems with superlinear power-like nonlinearities. One is interested, for example, in the existence or nonexistence, the uniqueness or multiplicity, the qualitative and asymptotic behavior of *entire solutions* (nontrivial solutions on $\mathbb{R}^n$), *large*

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solutions (solutions on bounded domains that blow up at the boundary), or solutions of the classical boundary-value problems, most notably the Dirichlet problem on bounded domains.

Higher-order analogues of these equations are of significant interest, both from the view-point of applications and on purely mathematical grounds (we refer the reader to a forthcoming monograph on polyharmonic boundary-value problems by Gazzola, Grunau, and Sweers). The biharmonic equation

$$\Delta^2 u = u|u|^{p-1} \tag{1.1}$$

is a prime example. While it combines many features of the two second-order equations mentioned above, it also exhibits interesting phenomena absent in the second-order case and poses much greater mathematical challenges, mostly due to the lack of a maximum principle.

Numerous known results for Eq. (1.1) and its second-order analogues, as well as the available methods of proof, depend on whether the growth of the nonlinearity is *subcritical*, *critical*, or *supercritical* in the sense that p is less than, equal to, or greater than the Sobolev critical exponent p^*, given by $p^* := (n+2m)/(n-2m)$ if $n > 2m$ and $p^* := \infty$ otherwise, with $m = 1$ in the second-order case, $m = 2$ in the fourth-order case. In particular, (1.1) has positive entire solutions if $p \geqslant p^*$ (see [2]), but no such solutions if $p < p^*$ (see [15]). The same holds for the second-order equation $-\Delta u = u|u|^{p-1}$ (see, for example, [26]).

In the subcritical case, a natural question is whether sign-changing entire solutions exist. For the second-order equation $-\Delta u = u|u|^{p-1}$, the answer is positive (see, for example, [1]). Somewhat surprisingly, the question appears to be open for the biharmonic equation (1.1). In the present paper, we give a positive answer in proving the existence and oscillatory nature of an entire radial solution of (1.1), unique up to scaling and symmetry, in the subcritical case. For completeness and contrast, we include in the following theorem the corresponding result for the critical/supercritical case, which is well known (see [2,5,15]), even in much more general situations (see [20,21]).

Theorem 1.1. *The problem*

$$\left(\partial_r^2 + \tfrac{n-1}{r}\partial_r\right)^2 u = u|u|^{p-1} \quad \text{in } (0,\infty), \qquad u(0) = 1 \tag{1.2}$$

has a unique solution $\bar{u} \in C^4([0,\infty),\mathbb{R})$. *Moreover,*

- *if* $p \geqslant p^*$, *then* $\bar{u}$ *is positive and decreasing with* $\bar{u}(r) \to 0$ *as* $r \to \infty$;
- *if* $p < p^*$, *then* $\bar{u}$ *is oscillatory with infinitely many zeros, all simple, and infinitely many critical points, all either positive local maxima or negative local minima.*

Remark 1.2. There is no loss of generality in imposing the condition $u(0) = 1$ in Problem (1.2). To see why, note that Eq. (1.1) has a trivial symmetry property (if u is a solution, then so is $-u$) and features a scaling-law: if $\lambda \in (0,\infty)$ and u is a solution, then so is the "rescaling" $x \mapsto \lambda^{4/(p-1)} u(\lambda x)$. Since entire radial solutions of (1.1) cannot vanish at the origin (as we shall see), it follows that any such solution is a rescaling of either $\bar{u}$ or $-\bar{u}$.

Remark 1.3. In the critical/supercritical case, the asymptotic behavior of the solution $\bar{u}$ of Theorem 1.1 is very well understood (see [2,4,5,9,15]). However, our understanding of the newly-found oscillatory solutions in the subcritical case is still far from complete. The detailed results of Section 5 go well beyond what is stated in Theorem 1.1, but do not answer, for example, the natural question whether these solutions vanish at infinity. In [14] we prove that, for $n = 1$, the solution $\bar{u}$ is a cosine-like periodic function. Work in progress indicates that, for $n > 1$, the amplitude of the oscillations of $\bar{u}$ decreases to zero, while the distance between consecutive zeros of $\bar{u}$ increases to infinity, both at specific rates depending on (and only on) p and n. This would generalize known results for the second-order equation $-\Delta u = u|u|^{p-1}$ (see [1]).

Besides comparison and scaling arguments, which will be used throughout the paper, the main ingredients of our proof of Theorem 1.1 are the following: (1) decay estimates for entire radial solutions of eventually constant sign (which imply the nonexistence of such solutions in the subcritical case by way of a Pohožaev-type identity); (2) energy estimates that allow us to show that oscillatory radial solutions cannot blow up (relevant only in the subcritical case); and (3) a meticulous tracking of the possible sign changes of u, $\partial_r u$, $(\partial_r^2 + \frac{n-1}{r}\partial_r)u$, and $\partial_r(\partial_r^2 + \frac{n-1}{r}\partial_r)u$ for a generic nontrivial radial solution u (a technique inspired by the work of Smillie [22] on tridiagonal systems of autonomous differential equations).

The main difficulty in finding and analyzing entire solutions of the biharmonic equation (1.1) is the fact, to be made precise in Section 4, that almost all its radial solutions blow up. Very different from the second-order equation $-\Delta u = u|u|^{p-1}$, whose radial solutions are all global (by a simple energy estimate), this feature of (1.1) is reminiscent of the second-order equation $\Delta u = u|u|^{p-1}$, whose nontrivial radial solutions all blow up (for example, by the classical blow-up criterion of Keller [10] and Osserman [17]).

In other words, generic radial solutions of (1.1) blow up, like those of $\Delta u = u|u|^{p-1}$, while entire radial solutions of (1.1) are nongeneric and behave like those of $-\Delta u = u|u|^{p-1}$. Our analysis yields precise information about both types of solutions. Like Theorem 1.1, the following result on large solutions appears to be new in the subcritical case; in the critical/supercritical case, it could be derived from results in [5,8] (see also Section 4 in [3]).

Theorem 1.4. *For every $\alpha \in \mathbb{R}$ the problem*

$$\left(\partial_r^2 + \tfrac{n-1}{r}\partial_r\right)^2 u = u|u|^{p-1} \quad \text{in } (0,1), \qquad u(0) = \alpha, \qquad u(1^-) = \infty \tag{1.3}$$

has a unique solution $u_\alpha^+ \in C^4([0,1),\mathbb{R})$. As $|\alpha| \to \infty$, the rescaling $r \mapsto \alpha^{-1}u_\alpha^+(|\alpha|^{-(p-1)/4}r)$ converges, uniformly on compact subintervals of $[0,\infty)$, to the unique solution $\bar{u}$ of Problem (1.2). Moreover,

- *if $p \geqslant p^*$, then u_α^+ does not change sign if $\alpha \geqslant 0$ and changes sign exactly once if $\alpha < 0$;*
- *if $p < p^*$, then the number of sign changes of u_α^+ grows without bound as $|\alpha| \to \infty$, successively attaining the values $0, 2, 4, \ldots$ as α increases from 0 to ∞ and the values $1, 3, 5, \ldots$ as α decreases from 0^- to $-\infty$.*

Remark 1.5. Due to the scaling-law mentioned in Remark 1.2, there is no loss of generality in posing Problem (1.3) on the interval $[0,1)$. In fact, every large radial solution of Eq. (1.1) is a rescaling of a large radial solution on the unit ball. We note that, due to symmetry, these solutions come in pairs. Specifically, for every $\alpha \in \mathbb{R}$, the function $u_\alpha^- := -u_{-\alpha}^+$ solves Problem (1.3) with $u(1^-) = -\infty$ instead of $u(1^-) = \infty$.

The assertions of Theorem 1.4 about the number of sign changes of large radial solutions, and the sharp contrast between the cases $p \geqslant p^*$ and $p < p^*$, are consequences of the observation that suitable rescalings of the solutions u_α^+ converge, as $|\alpha| \to \infty$, to the solution $\bar{u}$ of Theorem 1.1. Fig. 1 illustrates this observation. It shows, in a subcritical and a supercritical case, numerical approximations of $\bar{u}$ along with a number of large radial solutions. Via scaling and symmetry, each of the latter corresponds to one of the solutions u_α^+.

The graph on the left of Fig. 1 also hints at a way to construct radial solutions of the Dirichlet problem for Eq. (1.1), say, on the unit ball, that have a given number of sign changes. It is well known (see [6,7,12,19]) that the Dirichlet problem does not have any nontrivial radial solutions if $p \geqslant p^*$ (and the graph on the right of Fig. 1 illustrates why). If $p < p^*$, the existence of infinitely many radial solutions, including solutions of constant sign as well as sign-changing ones, can be established by variational methods, using a symmetric version of the mountain-pass theorem (see, for example, Chapter 2 of [7]). Our present analysis provides some additional information that seems to be otherwise unavailable, including the observation that, again, suitable rescalings of the solutions converge to the oscillatory entire solution $\bar{u}$ of Theorem 1.1. The relevance of this observation obviously depends on how well we succeed in understanding the nature and asymptotic behavior of $\bar{u}$ (see Remark 1.3).

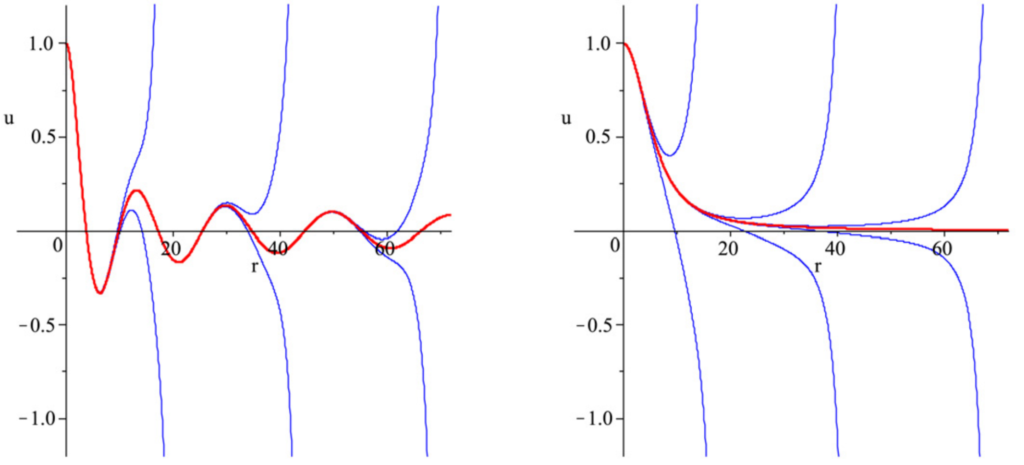


Fig. 1. Entire and large radial solutions for $p = 3$ and $n = 3$ (left), $n = 13$ (right).

Theorem 1.6. *Suppose that $p < p^*$. For every nonnegative integer k the problem*

$$\left(\partial_r^2 + \tfrac{n-1}{r}\partial_r\right)^2 u = u|u|^{p-1} \quad \text{in } (0,1), \qquad u(1) = u'(1) = 0 \tag{1.4}$$

has a unique pair of nontrivial solutions $\pm u_k \in C^4([0,1], \mathbb{R})$ with exactly k sign changes. The center value $\alpha_k := u_k(0)$ is positive and increases without bound as $k \to \infty$, while the rescaling $r \mapsto \alpha_k^{-1} u_k(\alpha_k^{-(p-1)/4} r)$ converges, uniformly on compact subintervals of $[0, \infty)$, to the unique solution $\bar{u}$ of Problem (1.2).

The paper is organized as follows. In Section 2, we introduce some necessary terminology, gather a number of largely known facts regarding the radial version of Eq. (1.1) and an equivalent first-order system, and establish decay estimates for global solutions of eventually constant sign. Section 3 provides a classification of the nonglobal solutions; one crucial step is to show that these cannot be oscillatory (that is, they change sign no more than finitely many times). In Section 4, we give a comprehensive description of the set of all radial solutions of (1.1). In particular, we obtain the existence and uniqueness, up to scaling and symmetry, of an entire radial solution and determine the exact multiplicity of large radial solutions. Section 5, the core of the paper, is concerned with the nodal properties of the solutions, especially in the subcritical case. Combining our results, we complete the proofs of Theorems 1.1, 1.4, and 1.6 in Section 6.

2. Preliminaries

In studying the radial version of the biharmonic equation (1.1) in $\mathbb{R}^n$, there is no need to restrict attention to integer dimensions n. Consequently, we will consider the ordinary differential equation

$$\left(\partial_r^2 + \tfrac{\mu}{r}\partial_r\right)^2 u = u|u|^{p-1} \tag{2.1}$$

for arbitrary $\mu \in \mathbb{R}_+$ and $p \in (1, \infty)$. Replacing n by $\mu + 1$, the notion of the Sobolev critical exponent p^* generalizes naturally, and a simple calculation shows that (2.1) is subcritical, critical, or supercritical, respectively, if μ is less than, equal to, or greater than $\mu^* := 2q+3$, where $q := 4/(p-1)$.

With $v_0 := u$, Eq. (2.1) is equivalent to the first-order system

$$\begin{cases} v_0' = v_1, & v_1' + \frac{\mu}{r} v_1 = v_2, \\ v_2' = v_3, & v_3' + \frac{\mu}{r} v_3 = v_0 |v_0|^{p-1}. \end{cases} \tag{2.2}$$

Definition 2.1. By a solution of (2.2), we mean an $\mathbb{R}^4$-valued function $\boldsymbol{v} = (v_0, v_1, v_2, v_3)$, defined and continuously differentiable on a maximal interval of the form $[0, r_\infty)$ with $r_\infty \in (0, \infty]$, solving the differential equations in (2.2) on $(0, r_\infty)$, and necessarily satisfying the symmetry/regularity conditions $v_1(0) = v_3(0) = 0$. We refer to v_0 and v_2 as the *even components*, to v_1 and v_3 as the *odd components* of $\boldsymbol{v}$. The pair of initial values $(v_0(0), v_2(0))$ is called the *starting-point* of $\boldsymbol{v}$, and we say that the solution starts at the point $(v_0(0), v_2(0))$. We refer to r_∞ as the *exit radius* of $\boldsymbol{v}$ and call the solution *global* if $r_\infty = \infty$, *explosive* if $r_\infty < \infty$; in the latter case, we also say that $\boldsymbol{v}$ blows up at r_∞.

The same terminology will be used in reference to solutions of Eq. (2.1), which are identified with the first components of solutions of (2.2). In addition, given a solution u of (2.1), the numbers $u(0)$ and $(\partial_r^2 + \frac{\mu}{r}\partial_r)u(0)$ will be called the *center value* and the *second center value* of u.

Remark 2.2. For every pair of initial values $(\alpha, \beta) \in \mathbb{R}^2$ there exists a unique solution of (2.2) starting at (α, β); if its exit radius is finite, the solution is unbounded. The solutions have the optimal regularity, determined by the degree of smoothness of the nonlinear term, $s \mapsto s|s|^{p-1}$, which is at least C^1. Thus, given a solution $\boldsymbol{v} = (v_0, v_1, v_2, v_3)$ with exit radius r_∞, we have $v_i \in C^{5-i}([0, r_\infty), \mathbb{R})$ for all $i \in \{0, 1, 2, 3\}$; in particular, the corresponding solution $u := v_0$ of (2.1) belongs to $C^5([0, r_\infty), \mathbb{R})$.

Solutions of (2.2) depend differentiably on their starting-points and the parameters μ and p. Suppose, for example, that $\boldsymbol{v}$ and $\boldsymbol{v}_k$, for $k \in \mathbb{N}$, are solutions with exit radii r_∞ and r_k, respectively. If $\boldsymbol{v}_k(0) \to \boldsymbol{v}(0)$ as $k \to \infty$, then $\liminf_{k\to\infty} r_k \geqslant r_\infty$ and $\boldsymbol{v}_k \to \boldsymbol{v}$ in the C^1-topology on compact subintervals of $[0, r_\infty)$. For the corresponding solutions u and u_k of Eq. (2.1), this means that $u_k \to u$ in the C^4-topology on compact subintervals of $[0, r_\infty)$. These assertions are analogous to standard results for regular ODE systems and can be proved in the same way.

Remark 2.3. The scaling-law, already mentioned in Remark 1.2, obviously extends to the system (2.2). In fact, given a solution $\boldsymbol{v}$ of (2.2) with exit radius r_∞ and a number $\lambda \in (0, \infty)$, let $v_i^{(\lambda)}(r) := \lambda^{q+i} v_i(\lambda r)$ for $r \in [0, r_\infty/\lambda)$, $i \in \{0, 1, 2, 3\}$, where $q := 4/(p-1)$. Then $\boldsymbol{v}^{(\lambda)} := (v_0^{(\lambda)}, v_1^{(\lambda)}, v_2^{(\lambda)}, v_3^{(\lambda)})$ is a solution of (2.2) with exit radius r_∞/λ. We call $\boldsymbol{v}^{(\lambda)}$ the λ*-rescaling* of $\boldsymbol{v}$. Likewise, $u^{(\lambda)} := v_0^{(\lambda)}$ is called the λ-rescaling of $u := v_0$.

If $\xi = (\alpha, \beta) \in \mathbb{R}^2$ and $\boldsymbol{v}$ is the solution of (2.2) starting at ξ, then $\boldsymbol{v}^{(\lambda)}$ is the solution starting at $\xi^{(\lambda)} := (\lambda^q \alpha, \lambda^{q+2}\beta)$. We call $\xi^{(\lambda)}$ the λ*-rescaling* of ξ and refer to the curve $\{\xi^{(\lambda)} \mid \lambda \in (0, \infty)\}$ as the *scaling-parabola* through ξ.

Remark 2.4. Given vectors $x, y \in \mathbb{R}^k$, for some positive integer k, we write $x \leqslant y$ or $y \geqslant x$ ($x < y$ or $y > x$) if the respective inequalities hold componentwise. If $x \leqslant y$ or $x \geqslant y$, we call the pair (x, y) *ordered*; if $x \geqslant 0$ ($x > 0$, $x \leqslant 0$, $x < 0$), we call x *nonnegative* (*positive, nonpositive, negative*).

The system (2.2) is quasimonotone and features a strong comparison principle, which can be proved in the same way as the analogous result for regular ODE systems (see, for example, [25]). Specifically, suppose that $\underline{\boldsymbol{v}}$ and $\bar{\boldsymbol{v}}$ are solutions, both defined on some interval $I \subset [0, \infty)$. If $\underline{\boldsymbol{v}}(r_0) \leqslant \bar{\boldsymbol{v}}(r_0)$ and $\underline{\boldsymbol{v}}(r_0) \neq \bar{\boldsymbol{v}}(r_0)$ for some $r_0 \in I$, then $\underline{\boldsymbol{v}}(r) < \bar{\boldsymbol{v}}(r)$ for all $r \in I$ with $r > r_0$. The same holds if $\underline{\boldsymbol{v}}$ and $\bar{\boldsymbol{v}}$ are, respectively, a subsolution and a supersolution of (2.2) on I, that is, functions in $C^1(I, \mathbb{R}^4)$ satisfying the differential inequalities obtained from the equations in (2.2) by replacing "=" by "$\leqslant$" and "$\geqslant$", respectively.

Since the constant 0 is a trivial solution of (2.2), the comparison principle implies, in particular, that if $\boldsymbol{v}$ is a nontrivial solution of (2.2) with exit radius r_∞, and if $\boldsymbol{v}(r_0) \geqslant 0$ ($\boldsymbol{v}(r_0) \leqslant 0$) for some $r_0 \in [0, r_\infty)$, then $\boldsymbol{v}(r) > 0$ ($\boldsymbol{v}(r) < 0$) for all $r \in (r_0, r_\infty)$.

Remark 2.5. Let $\boldsymbol{v} = (v_0, v_1, v_2, v_3)$ be a solution of the system (2.2) with exit radius r_∞. For every $r \in (0, r_\infty)$, the following Pohožaev-type identity is satisfied (see the beginning of this section for the definition of q and μ^*):

$$\begin{aligned}
&\frac{2(\mu-\mu^*)}{q+2}\int_0^r s^\mu \left|v_0(s)\right|^{p+1} ds \\
&\quad = r^{\mu+1}\biggl\{2v_1(r)v_3(r) + \frac{\mu-1}{r}\bigl(v_0(r)v_3(r) + v_1(r)v_2(r)\bigr) \\
&\qquad + \frac{2}{r}\bigl(v_1(r)v_2(r) - v_0(r)v_3(r)\bigr) - \left|v_2(r)\right|^2 - \frac{q}{q+2}\left|v_0(r)\right|^{p+1}\biggr\}.
\end{aligned} \tag{2.3}$$

This follows, for example, from the radial version of Eq. (2.16) in [16]. As it is well known (see [16,18, 19] for original work), the identity (2.3) precludes the existence of nontrivial solutions of the Dirichlet problem for Eq. (2.1) in the case $\mu \geqslant \mu^*$, as well as the existence of nontrivial global solutions with sufficiently fast decay at infinity in the case $\mu < \mu^*$.

Indeed, suppose that $\boldsymbol{v}$ is nontrivial and satisfies the Dirichlet condition at a point $r_0 \in (0, r_\infty)$, that is, $v_0(r_0) = v_1(r_0) = 0$. Then v_2 cannot vanish at r_0 (see, for example, Theorem 3.1 in [11]), and (2.3) implies that

$$\frac{2(\mu-\mu^*)}{q+2}\int_0^{r_0} s^\mu \left|v_0(s)\right|^{p+1} ds = -r_0^{\mu+1}\left|v_2(r_0)\right|^2 < 0.$$

This is impossible unless $\mu < \mu^*$.

Now suppose that $\boldsymbol{v}$ is nontrivial and global with

$$v_i(r) = O\bigl(r^{-q-i}\bigr) \quad \text{as } r \to \infty, \quad \text{for } i \in \{0,1,2,3\}. \tag{2.4}$$

Each term within the braces on the right-hand side of (2.3) is then $O(r^{-2q-4})$, and (2.3) implies that

$$\frac{2(\mu-\mu^*)}{q+2}\int_0^{r} s^\mu \left|v_0(s)\right|^{p+1} ds = O\bigl(r^{\mu-\mu^*}\bigr) \quad \text{as } r \to \infty.$$

This is impossible unless $\mu \geqslant \mu^*$.

The following lemma will allow us to preclude the existence of eventually positive global solutions of (2.1) in the case $\mu < \mu^*$, generalizing a well-known result by Mitidieri (see Theorem 3.5 in [16]). Like his, our proof is based on decay estimates and the Pohožaev-type identity (2.3).

Lemma 2.6. *Suppose that* (2.2) *has a global solution* $\boldsymbol{v} = (v_0, v_1, v_2, v_3)$ *such that* v_0 *is eventually positive and decreasing while* v_2 *is eventually negative and increasing. Then* $\mu \geqslant \mu^*$.

Proof. Let $\boldsymbol{v} = (v_0, v_1, v_2, v_3)$ be a global solution of (2.2) with v_0 positive and decreasing, v_2 negative and increasing on (r_0, ∞), for some $r_0 \in [0, \infty)$. Throughout the proof, let $v_4 := v_0|v_0|^{p-1}$ and $\hat{v}_i := r^\mu v_i$ for $i \in \{1,3\}$. Like v_0, the function v_4 is positive and decreasing on (r_0, ∞). Further, $v_i' = v_{i+1}$ for $i \in \{0,2\}$ and $\hat{v}_i' = r^\mu v_{i+1}$ for $i \in \{1,3\}$. Hence $\hat{v}_1$ is negative and decreasing, while $\hat{v}_3$ is positive and increasing on (r_0, ∞).

It follows that, for every $r \in (r_0, \infty)$,

$$r^{\mu} v_3(r) = \hat{v}_3(r_0) + \int_{r_0}^{r} s^{\mu} v_4(s)\, ds \geqslant \frac{v_4(r)}{\mu + 1}\bigl(r^{\mu+1} - r_0^{\mu+1}\bigr). \tag{2.5}$$

Clearly, $v_0(r)$ decreases to a nonnegative limit L as $r \to \infty$. Assuming L to be positive, (2.5) would imply that $v_2'(r) = v_3(r) \to \infty$ as $r \to \infty$, contradicting the assumption that $v_2 < 0$ on (r_0, ∞). Thus, $v_0(r) \to 0$ as $r \to \infty$. In a similar way, observing that, for every $r \in (r_0, \infty)$,

$$r^{\mu} v_1(r) = \hat{v}_1(r_0) + \int_{r_0}^{r} s^{\mu} v_2(s)\, ds \leqslant \frac{v_2(r)}{\mu + 1}\bigl(r^{\mu+1} - r_0^{\mu+1}\bigr), \tag{2.6}$$

we see that $v_2(r) \to 0$ as $r \to \infty$. Now, since v_0 vanishes at ∞, we have

$$-v_0(r) = \int_{r}^{\infty} v_1(s)\, ds = \int_{r}^{\infty} s^{-\mu} \hat{v}_1(s)\, ds \leqslant \hat{v}_1(r) \int_{r}^{\infty} s^{-\mu}\, ds$$

for all $r \in (r_0, \infty)$. It follows that $\mu > 1$ (else, the right-hand side of the last inequality would be $-\infty$) and that $-v_0(r) \leqslant r v_1(r)/(\mu - 1)$, that is,

$$r\bigl|v_1(r)\bigr| \leqslant (\mu - 1)\bigl|v_0(r)\bigr| \tag{2.7}$$

for all $r \in (r_0, \infty)$. Similarly, we have

$$-v_2(r) = \int_{r}^{\infty} v_3(s)\, ds = \int_{r}^{\infty} s^{-\mu} \hat{v}_3(s)\, ds \geqslant \hat{v}_3(r) \int_{r}^{\infty} s^{-\mu}\, ds$$

for all $r \in (r_0, \infty)$, which implies that

$$r\bigl|v_3(r)\bigr| \leqslant (\mu - 1)\bigl|v_2(r)\bigr| \tag{2.8}$$

for all $r \in (r_0, \infty)$. Now let $r_1 := 2^{1/(\mu+1)} r_0$. Then $r^{\mu+1} - r_0^{\mu+1} \geqslant r^{\mu+1}/2$ whenever $r \geqslant r_1$. Hence (2.5) and (2.6) yield

$$r\bigl|v_4(r)\bigr| \leqslant 2(\mu + 1)\bigl|v_3(r)\bigr| \tag{2.9}$$

and

$$r\bigl|v_2(r)\bigr| \leqslant 2(\mu + 1)\bigl|v_1(r)\bigr| \tag{2.10}$$

for all $r \in (r_1, \infty)$. From (2.7)–(2.10) we obtain

$$\begin{aligned} r^4\bigl|v_0(r)\bigr|^p = r^4\bigl|v_4(r)\bigr| &\leqslant 2(\mu+1) r^3\bigl|v_3(r)\bigr| \leqslant 2(\mu+1)(\mu-1) r^2\bigl|v_2(r)\bigr| \\ &\leqslant 4(\mu+1)^2(\mu-1) r\bigl|v_1(r)\bigr| \leqslant 4(\mu+1)^2(\mu-1)^2\bigl|v_0(r)\bigr| \end{aligned}$$

for all $r \in (r_1, \infty)$, which implies that $|v_0(r)| \leqslant [2(\mu^2 - 1)]^{q/2} r^{-q}$. Applying again (2.7), (2.10), and (2.8), we conclude that

$$v_0(r) = O\left(r^{-q}\right), \qquad v_1(r) = O\left(r^{-q-1}\right), \qquad v_2(r) = O\left(r^{-q-2}\right), \qquad v_3(r) = O\left(r^{-q-3}\right)$$

as $r \to \infty$, that is, $\boldsymbol{v}$ satisfies (2.4). As seen in Remark 2.5, this is impossible unless $\mu \geqslant \mu^*$. $\square$

Remark 2.7. As a by-product of the proof, we infer that any solution $\boldsymbol{v}$ of (2.2) satisfying the conditions of Lemma 2.6 must vanish at infinity with decay rates given by (2.4).

3. Classification of explosive solutions

In general, all one can say about explosive solutions of an ODE system is that they have at least one unbounded component. Different components may behave in different ways, and any component may be unbounded from below, or from above, or both; in the latter case, it oscillates about zero, more and more rapidly, with unbounded amplitude. Oscillatory blow-up of this kind occurs in higher-order versions of Eq. (2.1), specifically the equation $(\partial_r^2 + \frac{\mu}{r}\partial_r)^m u = u|u|^{p-1}$ with $m \in \mathbb{N}$ and $m \geqslant 3$, and it is the only possible kind of explosive behavior in the equation $(\partial_r^2 + \frac{\mu}{r}\partial_r)^m u = -u|u|^{p-1}$, for arbitrary $m \in \mathbb{N}$ (see [13, Section 3]).

In this section we will show that the explosive solutions of Eq. (2.1) are much better behaved and fit one of two possible simple descriptions. The crucial step is to rule out the existence of explosive solutions with unbounded oscillations. In the critical/supercritical case, that is, for $\mu \geqslant \mu^*$, this is not an issue, given the existence of global solutions of constant sign (see Section 5). The following lemma does the trick for any $\mu \in \mathbb{R}_+$.

Lemma 3.1. *Let $\boldsymbol{v} = (v_0, v_1, v_2, v_3)$ be a solution of (2.2) with exit radius r_∞ and define $e : [0, r_\infty) \to \mathbb{R}$ by*

$$e := \frac{1}{p+1}|v_0|^{p+1} + \frac{1}{2}|v_2|^2 - v_1 v_3.$$

Suppose that $r_1, r_2 \in (0, r_\infty)$ are critical points of the first component v_0, with $r_1 < r_2 \leqslant (\sqrt{2}+1)r_1$. Then $e(r_2) \leqslant e(r_1)$.

Proof. A straightforward calculation shows that

$$e'(r) = \frac{2\mu}{r} v_1(r) v_3(r) \tag{3.1}$$

for every $r \in (0, r_\infty)$. If $\mu = 0$, the function e is constant, and there is nothing left to prove. Suppose now that $\mu > 0$ and let $r_1, r_2 \in (0, r_\infty)$ be critical points of v_0 with $r_1 < r_2$. To facilitate the argument, set $u := v_0$ and replace v_1 and v_3 with the corresponding derivatives of u. Then

$$\begin{aligned}
\int_{r_1}^{r_2} \frac{1}{r} v_1 v_3 &= \int_{r_1}^{r_2} \frac{u'}{r}\left(u'' + \frac{\mu}{r}u'\right)' = \int_{r_1}^{r_2} \frac{u'}{r} u''' + \mu \int_{r_1}^{r_2} \frac{u'}{r}\left(\frac{u'}{r}\right)' \\
&= -\int_{r_1}^{r_2} \left(\frac{u'}{r}\right)' u'' + \mu \int_{r_1}^{r_2} \left(\frac{1}{2}\left|\frac{u'}{r}\right|^2\right)' = -\int_{r_1}^{r_2} \left(-\frac{1}{r^2}u' + \frac{1}{r}u''\right) u''.
\end{aligned}$$

From this and (3.1) we get

$$\frac{e(r_2)-e(r_1)}{2\mu}=\int_{r_1}^{r_2}\frac{1}{r^2}u'u''-\int_{r_1}^{r_2}\frac{1}{r}|u''|^2. \tag{3.2}$$

Integrating by parts and applying the mean-value theorem of integral calculus, we find a point $\rho\in[r_1,r_2]$ such that

$$\begin{aligned}\int_{r_1}^{r_2}\frac{1}{r^2}u'u''&=\int_{r_1}^{r_2}\frac{1}{r^2}\left(\frac{1}{2}|u'|^2\right)'=\int_{r_1}^{r_2}\frac{1}{r^3}|u'|^2\\&=\left|u'(\rho)\right|^2\int_{r_1}^{r_2}\frac{1}{r^3}=\frac{1}{2}\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right)\left|u'(\rho)\right|^2.\end{aligned}$$

Further,

$$\left|u'(\rho)\right|=\left|\int_{r_1}^{\rho}u''\right|\leqslant\int_{r_1}^{r_2}|u''|\leqslant\left(\int_{r_1}^{r_2}\frac{1}{r}|u''|^2\right)^{1/2}\left(\int_{r_1}^{r_2}r\right)^{1/2}$$

and hence

$$\left|u'(\rho)\right|^2\leqslant\frac{1}{2}\left(r_2^2-r_1^2\right)\int_{r_1}^{r_2}\frac{1}{r}|u''|^2.$$

Therefore,

$$\int_{r_1}^{r_2}\frac{1}{r^2}u'u''\leqslant\left(\frac{r_2^2-r_1^2}{2r_1r_2}\right)^2\int_{r_1}^{r_2}\frac{1}{r}|u''|^2,$$

and substituting this into (3.2) yields

$$\frac{e(r_2)-e(r_1)}{2\mu}\leqslant\left[\left(\frac{r_2^2-r_1^2}{2r_1r_2}\right)^2-1\right]\int_{r_1}^{r_2}\frac{1}{r}|u''|^2.$$

Assume now that $r_2\leqslant(\sqrt{2}+1)r_1$. A simple calculation shows that the factor in brackets is then nonpositive, proving that $e(r_2)\leqslant e(r_1)$. $\square$

Proposition 3.2. *Let $\boldsymbol{v}$ be an explosive solution of the system* (2.2)*. Then either all the components of $\boldsymbol{v}$ are eventually increasing and diverge to ∞, or all of them are eventually decreasing and diverge to $-\infty$.*

Proof. Let $\boldsymbol{v}=(v_0,v_1,v_2,v_3)$ be an explosive solution of (2.2) with exit radius r_∞. Suppose that v_0 is unbounded from below and from above. Then v_0 has a sequence $(r_k)_{k\in\mathbb{N}}$ of critical points in $(0,r_\infty)$, increasing with limit r_∞, such that $|v_0(r_k)|\to\infty$ as $k\to\infty$. Since r_∞ is finite, consecutive terms of the sequence (r_k) eventually satisfy the assumptions of Lemma 3.1, whence $(e(r_k))$ is eventually nonincreasing. However, $e(r_k)\geqslant|v_0(r_k)|^{p+1}/(p+1)$ for all $k\in\mathbb{N}$, whence $e(r_k)\to\infty$ as $k\to\infty$. The contradiction shows that v_0 cannot be unbounded both from below and from above.

Let $v_4 := v_0|v_0|^{p-1}$. Integrating the differential equations in (2.2) gives $v_i(r) = v_i(0) + \int_0^r v_{i+1}(s)\,ds$ for $i \in \{0, 2\}$ and $v_i(r) = \int_0^r (s/r)^\mu v_{i+1}(s)\,ds$ for $i \in \{1, 3\}$, for all $r \in (0, r_\infty)$. Since r_∞ is finite, it is clear that if v_{i+1} is bounded from below (from above), then so is v_i, for every $i \in \{0, 1, 2, 3\}$. Now, assuming that v_0 is bounded from below (from above), so is v_4; in view of the preceding observation, all the remaining components are then bounded from below (from above) as well.

Since $\boldsymbol{v}$ is explosive and hence unbounded, this leaves only two possible cases: either all components of $\boldsymbol{v}$ are bounded from below and unbounded from above, or all components of $\boldsymbol{v}$ are bounded from above and unbounded from below.

Consider the first case, that is, assume that all components of $\boldsymbol{v}$ are bounded from below and unbounded from above. We claim that then all components of $\boldsymbol{v}$ are eventually increasing and diverge to ∞. To prove this, fix $i \in \{0, 2\}$ and choose $M \in \mathbb{R}_+$ such that $v_{i+1} \geqslant -M$. Then $(v_i + Mr)' = v_{i+1} + M \geqslant 0$; hence the mapping $r \mapsto v_i(r) + Mr$ increases to a limit $L \in (-\infty, +\infty]$, which implies that $v_i(r) \to L - Mr_\infty$ as $r \to r_\infty$. If L were finite, v_i would be bounded from above, which is not the case; thus, $v_i(r) \to \infty$ as $r \to r_\infty$. Now fix $i \in \{1, 3\}$ and choose $M \in \mathbb{R}_+$ such that $v_{i+1} \geqslant -M$ (recall that if v_0 is bounded from below, so is v_4). Then

$$\left(r^\mu\left[v_i + \frac{M}{\mu+1}r\right]\right)' = \mu r^{\mu-1}\left[v_i + \frac{M}{\mu+1}r\right] + r^\mu\left[v_i' + \frac{M}{\mu+1}\right]$$
$$= r^\mu\left[v_i' + \frac{\mu}{r}v_i + M\right] = r^\mu[v_{i+1} + M] \geqslant 0;$$

hence the mapping $r \mapsto r^\mu[v_i(r) + \frac{M}{\mu+1}r]$ increases to a limit $L \in (-\infty, \infty]$, which implies that $v_i(r) \to Lr_\infty^{-\mu} - \frac{M}{\mu+1}r_\infty$ as $r \to r_\infty$. If L were finite, v_i would be bounded from above, which is not the case; thus, $v_i(r) \to \infty$ as $r \to r_\infty$.

This proves that all components of $\boldsymbol{v}$ diverge to ∞; in particular, all are eventually positive. Since $v_0' = v_1$ and $v_2' = v_3$, it follows that v_0 and v_2 are eventually increasing (and then, so is v_4). To verify that v_1 and v_3 are eventually increasing as well, fix $i \in \{1, 3\}$ and choose $r_0 \in (0, r_\infty)$ such that $v_{i+1}(r) \geqslant v_{i+1}(s)$ for all $r \in [r_0, r_\infty)$ and $s \in [0, r]$. (This is possible since v_{i+1} is eventually increasing and diverges to ∞.) For every $r \in [r_0, r_\infty)$, we then have

$$r^\mu v_i(r) = \int_0^r s^\mu v_{i+1}(s)\,ds \leqslant \frac{r^{\mu+1}}{\mu+1}v_{i+1}(r),$$

that is, $v_i(r) \leqslant \frac{r}{\mu+1}v_{i+1}(r)$, whence

$$v_i'(r) = v_{i+1}(r) - \frac{\mu}{r}v_i(r) \geqslant v_{i+1}(r) - \frac{\mu}{\mu+1}v_{i+1}(r) = \frac{1}{\mu+1}v_{i+1}(r).$$

Since v_{i+1} is eventually positive, this shows that v_i is eventually increasing, completing the proof of our claim for the first case.

In the second case, that is, if all the components of $\boldsymbol{v}$ are bounded from above and unbounded from below, symmetry implies that all of them are eventually decreasing and diverge to $-\infty$. $\square$

Definition 3.3. We call an explosive solution of (2.2) *positive-explosive* if all its components are eventually increasing and diverge to ∞, *negative-explosive* if all its components are eventually decreasing and diverge to $-\infty$. The same terminology will be used in reference to a solution of Eq. (2.1) if the corresponding solution of (2.2) has the respective property.

Corollary 3.4. *Every solution of* (2.1) *or* (2.2) *is either positive-explosive, or negative-explosive, or global.*

So far, we have not addressed the question of *existence* of explosive solutions. However, due to the superlinear growth of the nonlinearity, it is not difficult to show that eventually positive or eventually negative solutions of (2.1) cannot be global (see, for example, [24]), whence must be either positive-explosive or negative-explosive. The following lemma, a corollary to a more general result in [3], also guarantees continuity of the exit radius as a function of initial values.

Lemma 3.5.

(a) *If a nontrivial solution of* (2.2) *is nonnegative* (*nonpositive*) *at some point of its interval of existence, then it is positive-explosive* (*negative-explosive*).
(b) *The exit radius of the solutions of* (2.2) *is a continuous function of their starting-points.*

Proof. Suppose that a nontrivial solution $\boldsymbol{v}$ of (2.2) is nonnegative at some point of its interval of existence. In view of Remark 2.4, the solution is then eventually positive. Thus, Proposition 2.7 in [3] applies, proving that $\boldsymbol{v}$ is explosive, and in fact positive-explosive, by Proposition 3.2. The remainder of Part (a) follows by symmetry.

To prove Part (b), recall that, as a consequence of standard continuous-dependence results (see Remark 2.2), the exit radius of the solutions of (2.2) is a lower-semicontinuous function of initial data and parameters. Now suppose that $\boldsymbol{v}$ is a positive-explosive solution with exit radius r_∞. Again by Proposition 2.7 in [3], given $\bar{r} \in (r_\infty, \infty)$, any solution $\tilde{\boldsymbol{v}}$ of (2.2) with $\tilde{\boldsymbol{v}}(0)$ sufficiently close to $\boldsymbol{v}(0)$ blows up before $\bar{r}$. That is, the exit radius of positive-explosive solutions is upper-semicontinuous as a function of their starting-points. By symmetry, the same is true for negative-explosive solutions. Since every solution of (2.2) is either positive-explosive, or negative-explosive, or global, it follows that the exit radius of the solutions of (2.2) is an upper-semicontinuous, and hence continuous, function of their starting-points. $\square$

4. Existence and multiplicity of global and explosive solutions

In this section we give a comprehensive description of the set of all solutions of Eq. (2.1) or the equivalent system (2.2). Scaling-arguments, based on the observations in Remark 2.3, and monotonicity arguments, based on the strong comparison principle discussed in Remark 2.4, will be used throughout and play an important role.

Given a number $\alpha \in \mathbb{R}$, consider the solutions u of (2.1) with center value $u(0) = \alpha$. This family of solutions is parametrized by the second center value, $\beta = (\partial_r^2 + \frac{\mu}{r}\partial_r)u(0)$, and completely ordered with respect to this parameter, thanks to the comparison principle.

If $\alpha = 0$, it is clear that exactly one of these solutions, the trivial solution $u_0 = 0$, is global; the solutions above or below u_0, that is, the solutions with $\beta > 0$ or $\beta < 0$, are positive-explosive or negative-explosive, respectively, by Lemma 3.5. Since the solutions with $\beta > 0$ ($\beta < 0$) are just rescalings of the solution with $\beta = 1$ ($\beta = -1$), it is also clear that the exit radius of these solutions decreases continuously from ∞ to 0 as β increases from 0 to ∞ (or decreases from 0 to $-\infty$). Hence, for every $R \in (0, \infty)$, there is a unique pair of solutions $u_{0,R}^{\pm}$ with center value 0 and exit radius R; one is positive-explosive and above u_0, the other negative-explosive and below u_0.

We will show that analogous statements hold for arbitrary $\alpha \in \mathbb{R}$. In the critical or supercritical case, this is essentially known (see [5,8] or Section 4 of [3]); however, earlier proofs do not carry over to the subcritical case. In particular the existence and uniqueness, up to scaling and symmetry, of a nontrivial global solution appears to be a new result in the subcritical case. The fact that all solutions of (2.1) are either positive-explosive, or negative-explosive, or global (Corollary 3.4) is a key ingredient in the proof.

Theorem 4.1. *Let $\alpha \in \mathbb{R}$ and $R \in (0, \infty)$.*

(a) *There exists a unique global solution u_α of* (2.1) *with $u_\alpha(0) = \alpha$.*
(b) *There exists a unique pair of explosive solutions $u_{\alpha,R}^{\pm}$ of* (2.1) *with $u_{\alpha,R}^{\pm}(0) = \alpha$ and exit radius R.*
(c) *These solutions satisfy $u_{\alpha,R}^{-} < u_\alpha < u_{\alpha,R}^{+}$ on $(0, R)$. Further, $u_{\alpha,R}^{+}$ is eventually increasing and approaches ∞, while $u_{\alpha,R}^{-}$ is eventually decreasing and approaches $-\infty$.*

Remark 4.2. (a) For $\alpha \in (0,\infty)$, the global solution u_α is the $\alpha^{1/q}$-rescaling of u_1; also, $u_{-\alpha} = -u_\alpha$ and, of course, $u_0 = 0$. In particular, all nontrivial global solutions are obtained, via scaling and symmetry, from the global solution with center value 1.

(b) Regarding explosive solutions, symmetry yields $u^-_{\alpha,R} = -u^+_{-\alpha,R}$ for every $\alpha \in \mathbb{R}$ and $R \in (0,\infty)$. Also, $u^+_{\alpha,R}$ is the R^{-1}-rescaling of $u^+_{\alpha R^q,1}$. In particular, all explosive solutions are obtained, via scaling and symmetry, from the positive-explosive solutions with exit radius 1.

Theorem 4.1 is a consequence of a more general result about completely ordered families of solutions of the system (2.2), Theorem 4.6 below. The following terminology is quite natural and will facilitate the statements of three preparatory lemmas.

By an *ordered pair of solutions* of (2.2), we mean a pair of solutions $\underline{\boldsymbol{v}}$ and $\bar{\boldsymbol{v}}$ with $\underline{\boldsymbol{v}}(0) \leqslant \bar{\boldsymbol{v}}(0)$. As a consequence of the comparison principle, any such pair satisfies $\underline{\boldsymbol{v}} \leqslant \bar{\boldsymbol{v}}$ on $[0,R)$, where R is the minimum of the exit radii of $\underline{\boldsymbol{v}}$ and $\bar{\boldsymbol{v}}$. Assuming that $\underline{\boldsymbol{v}}(0) \neq \bar{\boldsymbol{v}}(0)$, we call $\underline{\boldsymbol{v}}$ the *lower solution*, $\bar{\boldsymbol{v}}$ the *upper solution*; by a solution *in between* $\underline{\boldsymbol{v}}$ and $\bar{\boldsymbol{v}}$, we mean a solution $\boldsymbol{v}$ of (2.2) with $\underline{\boldsymbol{v}}(0) \leqslant \boldsymbol{v}(0) \leqslant \bar{\boldsymbol{v}}(0)$ and $\boldsymbol{v}(0) \notin \{\underline{\boldsymbol{v}}(0), \bar{\boldsymbol{v}}(0)\}$. By the comparison principle, any such solution exists at least on the interval $[0,R)$ and satisfies $\underline{\boldsymbol{v}} < \boldsymbol{v} < \bar{\boldsymbol{v}}$ on $(0,R)$.

Lemma 4.3. *The system* (2.2) *cannot have an ordered pair of distinct global solutions.*

Proof. Suppose $\underline{\boldsymbol{v}}$ and $\bar{\boldsymbol{v}}$ are global solutions of (2.2) with $\underline{\boldsymbol{v}}(0) \leqslant \bar{\boldsymbol{v}}(0)$ and $\underline{\boldsymbol{v}}(0) \neq \bar{\boldsymbol{v}}(0)$. From the inequality $2^{p-1}(b|b|^{p-1} - a|a|^{p-1}) \geqslant (b-a)|b-a|^{p-1}$, valid for all $a, b \in \mathbb{R}$ with $b \geqslant a$, it follows that $\boldsymbol{w} := \frac{1}{2}(\bar{\boldsymbol{v}} - \underline{\boldsymbol{v}})$ is a supersolution of (2.2) on $[0,\infty)$. Specifically, the components of $\boldsymbol{w}$ satisfy the first three equations in (2.2) and the differential inequality obtained from the fourth equation by replacing "$=$" with "$\geqslant$". By the comparison principle, the solution $\boldsymbol{v}$ of (2.2) with $\boldsymbol{v}(0) = \boldsymbol{w}(0)$ stays below $\boldsymbol{w}$ and, thus, cannot be positive-explosive. Since $\boldsymbol{w}(0) \geqslant 0$ and $\boldsymbol{w}(0) \neq 0$, this contradicts Lemma 3.5(a). □

Lemma 4.4. *Given any ordered pair of distinct solutions of* (2.2) *with the same finite exit radius, every solution in between has a larger exit radius. Further, the upper solution is necessarily positive-explosive, the lower one negative-explosive.*

Proof. Suppose $\underline{\boldsymbol{v}}$ and $\bar{\boldsymbol{v}}$ are distinct solutions of (2.2) with $\underline{\boldsymbol{v}} \leqslant \bar{\boldsymbol{v}}$, blowing up at the same point $r_\infty \in (0,\infty)$, and let $\boldsymbol{v}$ be a solution in between. Then $\boldsymbol{v}$ cannot blow up before r_∞ and satisfies $\underline{\boldsymbol{v}} < \boldsymbol{v} < \bar{\boldsymbol{v}}$ on $(0, r_\infty)$.

Now suppose that $\boldsymbol{v}$ blows up exactly at r_∞. Fix a point $r_0 \in (0, r_\infty)$ and consider the rescalings $\boldsymbol{v}^{(\lambda)}$ of $\boldsymbol{v}$, for $\lambda \in (0,\infty)$, as defined in Remark 2.3. Clearly, $\boldsymbol{v}^{(\lambda)}(r_0) \to \boldsymbol{v}(r_0)$ as $\lambda \to 1$. Since $\underline{\boldsymbol{v}}(r_0) < \boldsymbol{v}(r_0) < \bar{\boldsymbol{v}}(r_0)$, we thus have $\underline{\boldsymbol{v}}(r_0) < \boldsymbol{v}^{(\lambda)}(r_0) < \bar{\boldsymbol{v}}(r_0)$ whenever λ is sufficiently close to 1. By the comparison principle, $\boldsymbol{v}^{(\lambda)}$ must then exist at least on $[0, r_\infty)$. However, $\boldsymbol{v}^{(\lambda)}$ blows up at r_∞/λ, leading to a contradiction if $\lambda > 1$. This proves that the exit radius of $\boldsymbol{v}$ is larger than r_∞.

Since every solution in between $\underline{\boldsymbol{v}}$ and $\bar{\boldsymbol{v}}$ exists beyond r_∞, the upper solution $\bar{\boldsymbol{v}}$ cannot be negative-explosive and so, due to Proposition 3.2, must be positive-explosive. Similarly, $\underline{\boldsymbol{v}}$ must be negative-explosive. □

Lemma 4.5. *Given any ordered pair of distinct solutions of* (2.2)*, the upper one is positive-explosive or the lower one is negative-explosive.*

Proof. Suppose that the lower solution is not negative-explosive; by Corollary 3.4, it is then either positive-explosive or global. In the first case, the upper solution is clearly positive-explosive as well. In the second case, the upper solution cannot be negative-explosive. By Lemma 4.3, it cannot be global either and, thus, must be positive-explosive. □

In reference to a function $\xi : \mathbb{R} \to \mathbb{R}^2$, terms like "nondecreasing," "bounded from above," or "bounded from below" are naturally understood with respect to the partial ordering of $\mathbb{R}^2$ (that is, componentwise).

Theorem 4.6. *Let $\xi : \mathbb{R} \to \mathbb{R}^2$ be continuous and one-to-one, nondecreasing, and neither bounded from above nor bounded from below. For $t \in \mathbb{R}$, denote by $\boldsymbol{v}^t$ the solution of* (2.2) *starting at $\xi(t)$, and let $\rho(t)$ be its exit radius.*

(a) *There exists a number $\bar{t} \in \mathbb{R}$ such that $\boldsymbol{v}^t$ is global if $t = \bar{t}$, positive-explosive if $t > \bar{t}$, and negative-explosive if $t < \bar{t}$.*
(b) *The function $\rho : \mathbb{R} \to (0, \infty]$ is continuous, increasing from 0 to ∞ on the interval $(-\infty, \bar{t})$ and decreasing from ∞ to 0 on the interval $(\bar{t}, \infty)$.*

Proof. We begin by showing that $\boldsymbol{v}^t$ is positive-explosive if t is large enough and that $\rho(t) \to 0$ as $t \to \infty$.

By assumption, both components of the vector function $\xi = (\xi_1, \xi_2)$ are nondecreasing, and at least one of them is unbounded from above and hence diverges to infinity. In case that $\xi_1(t) \to \infty$ as $t \to \infty$, let $\beta := \xi_2(0)$ and note that $\xi(t) \geqslant (\xi_1(t), \beta)$ for all $t \in \mathbb{R}_+$. In case that $\xi_2(t) \to \infty$ as $t \to \infty$, let $\alpha := \xi_1(0)$ and note that $\xi(t) \geqslant (\alpha, \xi_2(t))$ for all $t \in \mathbb{R}_+$. In view of the comparison principle, this shows that it suffices to prove our claim for the two special cases where the curve parametrized by ξ is either a horizontal line or a vertical line.

Consider the case of a horizontal line, with ξ given by $\xi(t) := (t, \beta)$ for $t \in \mathbb{R}$, for some $\beta \in \mathbb{R}$. For $t \in (0, \infty)$, let $\tilde{\xi}(t)$ denote the $t^{-1/q}$-rescaling of $\xi(t)$, as defined in Remark 2.3; that is, $\tilde{\xi}(t) := (1, t^{-(q+2)/q}\beta)$. Clearly, $\tilde{\xi}(t) \to (1, 0)$ as $t \to \infty$. By Lemma 3.5(a), the solution of (2.2) starting at $(1, 0)$ is positive-explosive; by continuous dependence on initial values, so is every solution starting sufficiently close to $(1, 0)$. It follows that the solution starting at $\tilde{\xi}(t)$ is positive-explosive if t is large enough. Moreover, by Lemma 3.5(b), the exit radius $\tilde{\rho}(t)$ of the solution starting at $\tilde{\xi}(t)$ converges, as $t \to \infty$, to the exit radius of the solution starting at $(1, 0)$, a finite number. Since $\xi(t)$ is the $t^{1/q}$-rescaling of $\tilde{\xi}(t)$, the solution $\boldsymbol{v}^t$ is nothing but the $t^{1/q}$-rescaling of the solution starting at $\tilde{\xi}(t)$ and therefore positive-explosive if t is large enough. Moreover, we have $\rho(t) = \tilde{\rho}(t)t^{-1/q}$, and since $\tilde{\rho}(t)$ converges to a finite number, $\rho(t)$ converges to 0 as $t \to \infty$.

Now consider the case of a vertical line, with ξ given by $\xi(t) := (\alpha, t)$ for $t \in \mathbb{R}$, for some $\alpha \in \mathbb{R}$. For $t \in (0, \infty)$, define $\tilde{\xi}(t) := (t^{-q/(q+2)}\alpha, 1)$; this is the $t^{-1/(q+2)}$-rescaling of $\xi(t)$. Clearly, $\tilde{\xi}(t) \to (0, 1)$ as $t \to \infty$, and the assertion follows with the same argument as in the previous case.

In conclusion, $\boldsymbol{v}^t$ is positive-explosive if t is large enough, and $\rho(t) \to 0$ as $t \to \infty$. By symmetry, $\boldsymbol{v}^t$ is negative-explosive if $-t$ is large enough, and $\rho(t) \to 0$ as $t \to -\infty$.

In the sequel we repeatedly use the fact that the solutions $\boldsymbol{v}^t$ are completely ordered with respect to t and depend continuously on t. Let $\bar{t}$ denote the infimum of those numbers $t \in \mathbb{R}$ such that $\boldsymbol{v}^t$ is positive-explosive. Then $\boldsymbol{v}^t$ is positive-explosive for all $t > \bar{t}$, while $\boldsymbol{v}^{\bar{t}}$ is not positive-explosive (else $\boldsymbol{v}^t$, with $t < \bar{t}$ and close enough to $\bar{t}$, would still be positive-explosive). Lemma 4.5 then implies that $\boldsymbol{v}^t$ is negative-explosive for all $t < \bar{t}$. Since $\boldsymbol{v}^{\bar{t}}$ cannot be negative-explosive (else $\boldsymbol{v}^t$, with $t > \bar{t}$ and close enough to $\bar{t}$, would still be negative-explosive), it must be global, by Corollary 3.4. This completes the proof of Part (a).

The continuity of ρ is clear, by Lemma 3.5(b), and we have already shown that $\rho(\bar{t}) = \infty$ and $\rho(t) \to 0$ as $t \to \pm\infty$. Finally, given an ordered pair of distinct positive-explosive solutions, the lower one cannot blow up before the upper one, and by Lemma 4.4, they cannot blow up at the same point either. This proves that ρ is decreasing on $(\bar{t}, \infty)$. By the same rationale, ρ is increasing on $(-\infty, \bar{t})$. The proof of the theorem is thus complete. □

The curve parametrized by ξ in Theorem 4.6 can, of course, be any straight line in $\mathbb{R}^2$ with a nonnegative direction vector; this includes horizontal as well as vertical lines. Proving Theorem 4.1 is now easy.

Proof of Theorem 4.1. Fix $\alpha \in \mathbb{R}$. For $\beta \in \mathbb{R}$, denote by $\boldsymbol{v}^\beta$ the solution of the system (2.2) starting at (α, β), and let $\rho(\beta)$ be its exit radius. Applying Theorem 4.6 with ξ given by $\xi(\beta) := (\alpha, \beta)$ for $\beta \in \mathbb{R}$, we find a number $\bar{\beta} \in \mathbb{R}$ such that $\boldsymbol{v}^\beta$ is global if $\beta = \bar{\beta}$, positive-explosive if $\beta > \bar{\beta}$, negative-explosive if $\beta < \bar{\beta}$. Part (a) of the theorem follows, with u_α the first component of $\boldsymbol{v}^{\bar{\beta}}$.

Now let $R \in (0, \infty)$. Since ρ is continuous, decreasing from ∞ to 0 on the interval $(\bar{\beta}, \infty)$ and increasing from 0 to ∞ on $(-\infty, \bar{\beta})$, there exist unique numbers $\beta^+ \in (\bar{\beta}, \infty)$ and $\beta^- \in (-\infty, \bar{\beta})$ such that $\rho(\beta^\pm) = R$. Part (b) of the theorem follows, with $u^\pm_{\alpha,R}$ the first component of $\boldsymbol{v}^{\beta^\pm}$. Part (c) is clear, by the comparison principle and Proposition 3.2. □

Corollary 4.7. *For $\alpha \in \mathbb{R}$ and $R \in (0, \infty)$, let $u^+_{\alpha,R}$ ($u^-_{\alpha,R}$) denote the unique positive-explosive (negative-explosive) solution of* (2.1) *with center value α and exit radius R. As $|\alpha| \to \infty$, the rescaling $r \mapsto \alpha^{-1} u^\pm_{\alpha,R}(|\alpha|^{-1/q} r)$ converges, in the C^4-topology on compact subintervals of $[0, \infty)$, to the unique global solution of* (2.1) *with center value* 1.

Proof. For $\alpha \in \mathbb{R} \setminus \{0\}$ and $R \in (0, \infty)$, the function $r \mapsto \alpha^{-1} u^\pm_{\alpha,R}(|\alpha|^{-1/q} r)$ is a solution of (2.1) with center value 1 and exit radius $R|\alpha|^{1/q}$; let $\beta(\alpha)$ be its second center value. Denoting by $\rho(t)$, for $t \in \mathbb{R}$, the exit radius of the solution starting at $(1, t)$, we then have $\rho(\beta(\alpha)) = R|\alpha|^{1/q} \to \infty$ as $|\alpha| \to \infty$. By the monotonicity properties of ρ (see Theorem 4.6(b)), it follows that, as $|\alpha| \to \infty$, $\beta(\alpha)$ converges to the second center value of the unique global solution with center value 1. The assertion of the corollary is thus a consequence of continuous dependence on initial values (see Remark 2.2). □

The significance of Corollary 4.7 is that it allows us to infer information about the behavior of the explosive solutions of Eq. (2.1) from the behavior of the nontrivial global solutions. This will be pursued in Section 5. In closing the present section, we give a geometric interpretation of our results regarding the structure of the set of all solutions.

Let $\mathcal{H}_\infty$ be the subset of $\mathbb{R}^2$ consisting of the starting-points of the global solutions of (2.1). Further, for $R \in (0, \infty)$, let $\mathcal{H}^+_R$ ($\mathcal{H}^-_R$) denote the set of all points in $\mathbb{R}^2$ that are starting-points of positive-explosive solutions (negative-explosive solutions) with exit radius R. By Corollary 3.4, these pairwise disjoint sets form a partition of $\mathbb{R}^2$, that is,

$$\mathbb{R}^2 = \mathcal{H}_\infty \cup \bigcup_{R \in (0,\infty)} \left(\mathcal{H}^+_R \cup \mathcal{H}^-_R\right).$$

Each of the sets is unordered (that is, does not contain any ordered pair of distinct points) and divides $\mathbb{R}^2$ into two half-spaces, above and below the set (all with respect to the componentwise ordering of $\mathbb{R}^2$). Given any two of the sets (say, $\mathcal{A}$ and $\mathcal{B}$), one of them is below the other (in symbols, $\mathcal{A} < \mathcal{B}$ or $\mathcal{B} < \mathcal{A}$). In fact, given $R, R' \in (0, \infty)$ with $R < R'$, we have

$$\mathcal{H}^-_R < \mathcal{H}^-_{R'} < \mathcal{H}_\infty < \mathcal{H}^+_{R'} < \mathcal{H}^+_R.$$

All of this follows from Theorem 4.6, which shows that every continuous completely ordered curve in $\mathbb{R}^2$ that is neither bounded from above nor bounded from below intersects each of the sets $\mathcal{H}^-_R$, $\mathcal{H}^-_{R'}$, $\mathcal{H}_\infty$, $\mathcal{H}^+_{R'}$, $\mathcal{H}^+_R$ in exactly one point, and in this order.

By Remark 4.2(a), $\mathcal{H}_\infty$ consists of the origin and a symmetric pair of scaling-parabolae (see Remark 2.3 for terminology and notation), namely, the scaling-parabolae through the points $\pm(1, \bar{\beta})$, where $\bar{\beta}$ is the second center value of the unique global solution of (2.1) with center value 1. In other words, $\mathcal{H}_\infty$ is the graph of the odd continuous function $\beta_\infty : \mathbb{R} \to \mathbb{R}$, defined by $\beta_\infty(\alpha) := \alpha|\alpha|^{2/q}\bar{\beta}$ for $\alpha \in \mathbb{R}$. Note that, by Lemma 3.5(a), $\bar{\beta}$ is a negative number, so that β_∞ is decreasing.

By Remark 4.2(b), the set $\mathcal{H}^-_R$, for any $R \in (0, \infty)$, is symmetric to $\mathcal{H}^+_R$, and $\mathcal{H}^+_R$ is a rescaling of $\mathcal{H}^+_1$; in fact,

$$\mathcal{H}^-_R = -\mathcal{H}^+_R \quad \text{and} \quad \mathcal{H}^+_R = \left(\mathcal{H}^+_1\right)^{(1/R)} := \left\{\xi^{(1/R)} \mid \xi \in \mathcal{H}^+_1\right\}.$$

Further, since $\mathcal{H}^\pm_R$ is unordered, it is the graph of a decreasing function $\beta^\pm_R : \mathbb{R} \to \mathbb{R}$, necessarily continuous (again by Theorem 4.6).

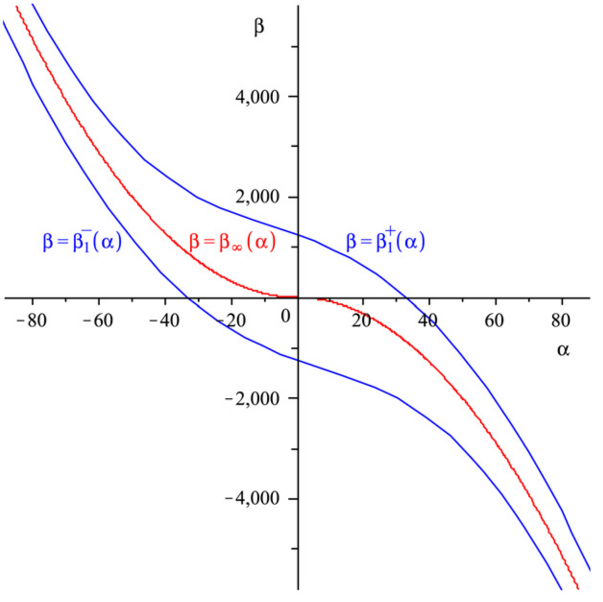


Fig. 2. The sets $\mathcal{H}_\infty$ and $\mathcal{H}_1^\pm$ for $p = 3$ and $\mu = 2$.

Note that $\beta_R^\pm(\alpha)$, for $\alpha \in \mathbb{R}$ and $R \in (0, \infty)$, is nothing but the second center value of the solution $u_{\alpha,R}^\pm$ of Theorem 4.1. The second center value of the rescaling $r \mapsto \alpha^{-1} u_{\alpha,R}^\pm(|\alpha|^{-1/q} r)$ is given by $\alpha^{-1}|\alpha|^{-2/q}\beta_R^\pm(\alpha)$. As $|\alpha| \to \infty$, this rescaling converges to the global solution with center value 1, by Corollary 4.7. It follows that $\alpha^{-1}|\alpha|^{-2/q}\beta_R^\pm(\alpha) \to \bar{\beta}$ or, equivalently, $\beta_R^\pm(\alpha)/\beta_\infty(\alpha) \to 1$ as $|\alpha| \to \infty$. Geometrically, this means that the sets $\mathcal{H}_\infty$ and $\mathcal{H}_R^\pm$ "merge at infinity" in the second and fourth quadrants. Fig. 2 gives an illustration, based on the numerical computation of $\bar{\beta}$ and a large number of starting-points of explosive solutions in a special case.

The global structure of the set of all solutions of Eq. (2.1), captured by the above partitioning of the "phase space" $\mathbb{R}^2$, has an analogue for the polyharmonic equation $(\partial_r^2 + \frac{\mu}{r}\partial_r)^m u = u|u|^{p-1}$ with arbitrary $m \in \mathbb{N}$. The notion of positive-explosive/negative-explosive solutions generalizes naturally, and the sets $\mathcal{H}_R^\pm$, defined in analogy with the case $m = 2$, and $\mathcal{H}_\infty := \mathbb{R}^m \setminus \bigcup_{R \in (0,\infty)}(\mathcal{H}_R^+ \cup \mathcal{H}_R^-)$ are $(m-1)$-dimensional unordered manifolds in $\mathbb{R}^m$. However, if $m \geqslant 3$, the solutions starting on $\mathcal{H}_\infty$ are generally not global, but oscillatory-explosive in the sense mentioned at the beginning of Section 3. We refer to [13] for details.

5. Nodal properties of global and explosive solutions

It is well known that in the critical or supercritical case, that is, for $\mu \geqslant \mu^* := 2q + 3$ with $q := 4/(p-1)$, Eq. (2.1) has positive global solutions (see, for example, [2]). Based on what we proved in Section 4, it is easy to see why; we provide the argument here to make the proof of our main result, Theorem 5.4, self-contained.

Lemma 5.1. *Suppose that $\mu \in [\mu^*, \infty)$. Then every nontrivial global solution of Eq.* (2.1) *is either positive throughout or negative throughout.*

Proof. Due to symmetry and the scaling-law, it suffices to consider the global solution of (2.1) with center value 1, that is, the first component of the global solution of (2.2) starting on the line $\{(1, \beta) \mid \beta \in \mathbb{R}\}$. For $\beta \in \mathbb{R}$, let $\boldsymbol{v}^\beta = (v_0^\beta, v_1^\beta, v_2^\beta, v_3^\beta)$ denote the solution starting at $(1, \beta)$, and let $\bar{\beta}$ be the unique (necessarily negative) real number such that $\boldsymbol{v}^{\bar{\beta}}$ is global.

Define $\beta_0 := \inf\{\beta \in \mathbb{R} \mid v_0^\beta \geqslant 0 \text{ throughout}\}$. Clearly, $\beta_0 \in [\bar\beta, 0)$, and $\boldsymbol{v}^{\beta_0}$ is the smallest among the solutions $\boldsymbol{v}^\beta$ whose first component is nonnegative throughout. We claim that, if $\mu \geqslant \mu^*$, then $\boldsymbol{v}^{\beta_0}$ is global. Assume the contrary. Then $\boldsymbol{v}^{\beta_0}$ is positive-explosive, and $v_0^{\beta_0}$ attains a global minimum, necessarily with value 0 (else, solutions $\boldsymbol{v}^\beta$ with $\beta < \beta_0$ and sufficiently close to β_0 would still have no sign change in the first component, contradicting the definition of β_0). At the minimum point r_0 of $v_0^{\beta_0}$, the solution $\boldsymbol{v}^{\beta_0}$ satisfies the Dirichlet condition, that is, $v_0^{\beta_0}(r_0) = v_1^{\beta_0}(r_0) = 0$. By Remark 2.5, this is impossible unless $\mu < \mu^*$. It follows that, if $\mu \geqslant \mu^*$, then $\bar\beta = \beta_0$ and $v_0^{\bar\beta} = v_0^{\beta_0} \geqslant 0$ throughout. □

We note in passing that much more is known about the nontrivial global solutions of (2.1) in the critical/supercritical case. For $\mu = \mu^*$, these solutions are known explicitly (see [15] or [23]). For $\mu > \mu^*$, they all have, up to symmetry, the same asymptotic profile, given by an explicitly known singular solution, and either oscillate about this profile or approach it monotonically, depending on whether $\mu < \mu^{**}$ or $\mu \geqslant \mu^{**}$, for some number $\mu^{**} \in (\mu^*, \infty)$ (see [4,5,9]).

By contrast, if $\mu < \mu^*$, the only a priori information we have regarding the nature of the nontrivial global solutions of (2.1) is that they must be sign-changing (see, for example, Theorem 1.4 in [15]). We will show that any such solution is, in fact, oscillatory, with a cyclic pattern of sign changes in the four components of the corresponding solution of (2.2).

Theorem 5.2. *If $\mu \in [0, \mu^*)$, then every nontrivial global solution of* (2.1) *is oscillatory, with infinitely many zeros, all simple, and infinitely many critical points, all either positive local maxima or negative local minima.*

Theorem 5.2 follows from a much more detailed result about the sequence of sign changes in the four components of an arbitrary nontrivial solution of (2.2), for arbitrary $\mu \in \mathbb{R}_+$. Note that each component of such a solution has at most isolated zeros; thus, the set of all points where at least one component changes sign has no limit point in the solution's interval of existence and allows an increasing enumeration. Some additional notation will be useful.

Definition 5.3. Let $\boldsymbol{v} = (v_0, v_1, v_2, v_3)$ be a nontrivial solution of (2.2). Given $i, j \in \{0, 1, 2, 3\}$ with $i \neq j$, we write $\langle i, j\rangle$ to denote a pair of consecutive sign changes where first v_i changes sign, then v_j changes sign. We write $\langle i/j\rangle$ to denote a pair of consecutive or simultaneous sign changes where both v_i and v_j change sign, in either order or simultaneously. By a *nodal cycle*, denoted by $\langle 0/2, 1/3\rangle$, we mean a sequence of four consecutive or simultaneous sign changes, where first the even components v_0 and v_2 change sign (in either order or simultaneously), then the odd components v_1 and v_3 change sign (in either order or simultaneously).

Theorem 5.4. *Every nontrivial solution $\boldsymbol{v}$ of* (2.2) *has one and only one of the following four properties*:

(a) *$\boldsymbol{v}$ is global and has no sign changes in any component*;
(b) *$\boldsymbol{v}$ is explosive and has no sign changes in any component*;
(c) *$\boldsymbol{v}$ is global, and the sequence of its sign changes consists of an infinite number of consecutive nodal cycles*;
(d) *$\boldsymbol{v}$ is explosive, and the sequence of its sign changes consists of a nonnegative number of consecutive nodal cycles, followed by exactly one of the sequences $\langle 0, 3\rangle$, $\langle 2, 1\rangle$, $\langle 0/2, 1, 0\rangle$, or $\langle 0/2, 3, 2\rangle$.*

Moreover, if $\mu < \mu^*$, then (a) *is impossible. If $\mu \geqslant \mu^*$, then* (c) *is impossible, and the sequence of sign changes of any solution satisfying* (d) *is either $\langle 0, 3\rangle$ or $\langle 2, 1\rangle$.*

Proof. Let $\boldsymbol{v} = (v_0, v_1, v_2, v_3)$ be a nontrivial solution of (2.2) with exit radius r_∞ and let S be the (possibly empty) sequence of its sign changes. Throughout the proof, let $\hat v_i := r^\mu v_i$, for $i \in \{1, 3\}$, and $v_4 := v_0|v_0|^{p-1}$. Sign, zeros, and monotonicity of v_4 clearly coincide with sign, zeros, and monotonicity of v_0. We note that $v_i' = v_{i+1}$ if $i \in \{0, 2\}$, $\hat v_i' = r^\mu v_{i+1}$ if $i \in \{1, 3\}$; hence the sign of v_{i+1} determines the monotonicity of v_i for $i \in \{0, 2\}$, of $\hat v_i$ for $i \in \{1, 3\}$. It follows that no two consecutive components of the $\mathbb{R}^5$-valued function $(v_0, v_1, v_2, v_3, v_4)$ can change sign simultaneously (even though consecutive components may vanish simultaneously). We also note that if $r_0 \in [0, r_\infty)$ and $v_i(r_0) = 0$ for some $i \in \{0, 1, 2, 3\}$, then r_0 is a simple zero of v_i if and only if $v_{i+1}(r_0) \neq 0$.

Initially, we have $v_1(0) = v_3(0) = 0$. If $v_0(0)v_2(0) \geqslant 0$, the solution starts and stays in the nonnegative cone $\mathbb{R}^4_+$ or the nonpositive cone $\mathbb{R}^4_-$ of $\mathbb{R}^4$; it then satisfies Condition (b) of the theorem. Therefore, and because of symmetry, we can and will assume that $v_0(0) > 0$ and $v_2(0) < 0$.

The "nodal state" of $\boldsymbol{v}$ at $r = 0$ is then described by Diagram 1. Since $v_0(0)$ is positive, so is $v_4(0)$, and $\hat{v}_3$ is increasing at 0. Thus, $\hat{v}_3$ is positive immediately after the start (that is, in some interval $(0, \delta)$ with $\delta > 0$), and so is v_3. Similarly, since $v_2(0)$ is negative, v_1 is negative immediately after the start. Therefore, the state of $\boldsymbol{v}$ immediately after the start is described by Diagram 2.

Diagram 1

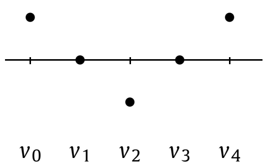
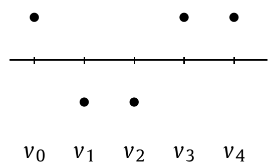


Diagram 2



If no component of $\boldsymbol{v}$ ever vanishes henceforth, then v_0 is positive and decreasing throughout, while v_2 is negative and increasing throughout; in particular, $\boldsymbol{v}$ is a global solution and satisfies Condition (a) of the theorem. As the remainder of the proof will show, this is the one and only way for Condition (a) to arise. By Lemma 2.6, it cannot arise at all if $\mu < \mu^*$.

From now on, suppose that the state described in Diagram 2 does not persist, that is, at least one component of $\boldsymbol{v}$ vanishes at some point. As long as v_0 is positive, so is v_4; thus $\hat{v}_3$ increases and is positive, and so is v_3. As long as v_2 is negative, $\hat{v}_1$ decreases and is negative, and so is v_1. Therefore, the first zero must occur in v_0 or v_2. Three possible scenarios arise.

Case 1. The components v_0 and v_2 vanish simultaneously, say at $\bar{r} \in (0, r_\infty)$. In this case, $\bar{r}$ is a simple zero of both v_0 and v_2 (since $v_1(\bar{r}) < 0$ and $v_3(\bar{r}) > 0$); both the even components change sign at $\bar{r}$, and the solution enters the state described in Diagram 3.

Diagram 3

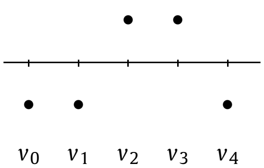


Diagram 3a

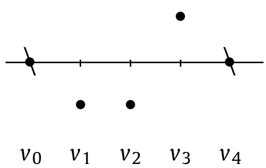
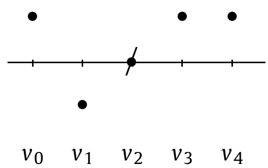


Diagram 3b



Case 2. The component v_0 vanishes first, say at $\bar{r} \in (0, r_\infty)$, while v_2 is still negative. At $\bar{r}$, a simple zero of v_0 (since $v_1(\bar{r}) < 0$), the solution is then in the state described in Diagram 3a, with v_0 (and v_4) changing sign from positive to negative, as indicated by the diagonal line crossing the corresponding bullet. Suppose no component ever vanishes henceforth. Again, the solution is then necessarily global. Since v_0 is negative and decreasing on $(\bar{r}, \infty)$, so is v_4 and then also $r^\mu v_4 = \hat{v}_3'$. Clearly, the fact that $\hat{v}_3'$ is eventually negative and decreasing implies that $\hat{v}_3(r) \to -\infty$ as $r \to \infty$. But this is impossible, since v_3 (whence $\hat{v}_3$) remains positive. It follows that there must be a further zero, and only v_2 and v_3 are candidates to vanish next. If v_3 vanishes before or simultaneously with v_2, say at $\hat{r} \in (\bar{r}, r_\infty)$, then $\hat{r}$ is a simple zero of v_3 (since $v_0(\hat{r}) < 0$) and, possibly, a double zero of v_2; v_3 changes sign from positive to negative at $\hat{r}$, while v_2 remains nonpositive and the components v_0 and v_1 are negative. Thus, the solution enters $\mathbb{R}^4_-$; no further sign changes occur, and the solution blows up. In this case, $\boldsymbol{v}$ satisfies condition (d) of the theorem with $S = \langle 0, 3 \rangle$. On the other hand, if v_2 vanishes first, while v_3 is still positive, then v_2 has a simple zero, changes sign, and the solution enters the state described in Diagram 3.

Case 3. The component v_2 vanishes first, say at $\bar{r} \in (0, r_\infty)$, while v_0 is still positive. At $\bar{r}$, a simple zero of v_2 (since $v_3(\bar{r}) > 0$), the solution is then in the state described in Diagram 3b, with v_2 changing sign from negative to positive. Suppose no component ever vanishes henceforth. Again, the solution is necessarily global. Since v_2 is positive and increasing in $(\bar{r}, \infty)$, so is $r^\mu v_2 = \hat{v}_1'$. This implies that $\hat{v}_1(r) \to \infty$ as $r \to \infty$, which is impossible since v_1 remains negative. It follows that a further zero must occur, and only v_0 and v_1 are candidates to vanish next. If v_1 vanishes before or simultaneously with v_0, say at $\hat{r} \in (\bar{r}, r_\infty)$, then $\hat{r}$ is a simple zero of v_1 (since $v_2(\hat{r}) > 0$) and, possibly, a double zero of v_0; v_1 changes sign from negative to positive at $\hat{r}$, while v_0 remains nonnegative and the components v_2 and v_3 are positive. Hence the solution enters $\mathbb{R}^4_+$, and $\boldsymbol{v}$ satisfies Condition (d) with $S = \langle 2, 1 \rangle$. If v_0 vanishes first, while v_1 is still negative, then v_0 has a simple zero, changes sign, and the solution enters the state described in Diagram 3.

Summing up, either the solution satisfies Condition (d) with $S = \langle 0, 3 \rangle$ or $S = \langle 2, 1 \rangle$; or, starting from the state described in Diagram 2, the solution undergoes exactly two sign changes, in the even components v_0 and v_2, while v_1 and v_3 never vanish, and enters the state described in Diagram 3.

As shown in the analysis of Case 2 above, v_3 cannot be eventually positive while v_0 is eventually negative and decreasing. Thus, the state described in Diagram 3 cannot persist; a further zero must occur, and the only candidates to vanish next are v_1 and v_3. Again, three possible scenarios arise.

Case 1′. The components v_1 and v_3 vanish simultaneously, say at $\bar{r} \in (0, r_\infty)$. In this case, $\bar{r}$ is a simple zero of both v_1 and v_3 (since $v_2(\bar{r}) > 0$ and $v_0(\bar{r}) < 0$); both the odd components change sign at $\bar{r}$, and the solution enters the state described in Diagram 4.

Diagram 4

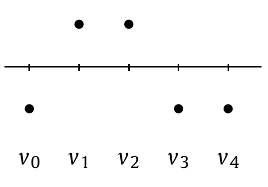


Diagram 4a

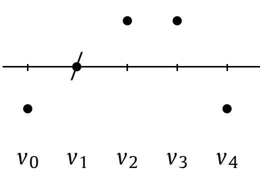
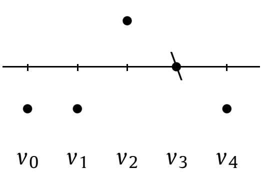


Diagram 4b



Case 2′. The component v_1 vanishes first, say at $\bar{r} \in (0, r_\infty)$, while v_3 is still positive. At $\bar{r}$, a simple zero of v_1 (since $v_2(\bar{r}) > 0$), the solution is then in the state described in Diagram 4a, with v_1 changing sign from negative to positive. Suppose no component ever vanishes henceforth. Then the solution is global, with $v_1 = v_0'$ positive and increasing on $(\bar{r}, \infty)$. This implies that $v_0(r) \to \infty$ as $r \to \infty$, contradicting the assumption that v_0 remains negative. It follows that a further zero must occur, and only v_0 and v_3 are candidates to vanish next. If v_0 vanishes before or simultaneously with v_3, say at $\hat{r} \in (\bar{r}, r_\infty)$, then $\hat{r}$ is a simple zero of v_0 (since $v_1(\hat{r}) > 0$) and, possibly, an even (though not double) zero of v_3; v_0 changes sign from negative to positive at $\hat{r}$, while v_3 remains nonnegative and the components v_1 and v_2 are positive. Hence the solution enters $\mathbb{R}^4_+$ and satisfies Condition (d) with $S = \langle 0/2, 1, 0 \rangle$. If v_3 vanishes first, while v_0 (whence v_4) is still negative, then v_3 has a simple zero, changes sign, and the solution enters the state described in Diagram 4.

Case 3′. The component v_3 vanishes first, say at $\bar{r} \in (0, r_\infty)$, while v_1 is still negative. At $\bar{r}$, a simple zero of v_3 (since $v_4(\bar{r}) < 0$), the solution is then in the state described in Diagram 4b, with v_3 changing sign from positive to negative. Suppose no component ever vanishes henceforth. Then the solution is global, with $v_3 = v_2'$ negative and decreasing on $(\bar{r}, \infty)$. This implies that $v_2(r) \to -\infty$ as $r \to \infty$, contradicting the assumption that v_2 remains positive. It follows that a further zero must occur, and only v_1 and v_2 are candidates to vanish next. If v_2 vanishes before or simultaneously with v_1, say at $\hat{r} \in (\bar{r}, r_\infty)$, then $\hat{r}$ is a simple zero of v_2 (since $v_3(\hat{r}) < 0$) and, possibly, a double zero of v_1; v_2 changes sign from positive to negative at $\hat{r}$, while v_1 remains nonpositive and the components v_0 and v_3 are negative. Hence the solution enters $\mathbb{R}^4_-$ and satisfies Condition (d) with

$S = \langle 0/2, 3, 2\rangle$. If v_1 vanishes first, while v_2 is still positive, then v_1 has a simple zero, changes sign, and the solution enters the state described in Diagram 4.

Summing up, either the solution satisfies Condition (d) with S given by $\langle 0/2, 1, 0\rangle$ or $\langle 0/2, 3, 2\rangle$; or, starting from the state described in Diagram 3, the solution undergoes exactly two sign changes, in the odd components v_1 and v_3, while v_0 and v_2 never vanish, and enters the state described in Diagram 4. At this stage, the solution has completed a nodal cycle.

The state described in Diagram 4 is symmetric to the one in Diagram 2, with the signs of all components of $\boldsymbol{v}$ reversed. If no component ever vanished again, v_0 would be eventually negative and increasing, v_2 eventually positive and decreasing, and the solution would be global. Applying Lemma 2.6 to the solution $-\boldsymbol{v}$, we see that this is impossible if $\mu < \mu^*$. But it is also impossible if $\mu \geqslant \mu^*$, since nontrivial global solutions, in this case, have no sign changes at all in the first component (here is where we need Lemma 5.1). It follows that at least one component of $\boldsymbol{v}$ must vanish again at some point, and the same possible scenarios as before ensue, albeit with the signs of all components of $\boldsymbol{v}$ reversed. Specifically, either the solution is explosive and S consists of a full nodal cycle followed by exactly one of the "terminal sequences" $\langle 0, 3\rangle$, $\langle 2, 1\rangle$, $\langle 0/2, 1, 0\rangle$, or $\langle 0/2, 3, 2\rangle$; or the solution completes a second nodal cycle and re-enters the state described in Diagram 2. In the latter case, the same rationale as before shows that this state cannot persist, whether $\mu < \mu^*$ or $\mu \geqslant \mu^*$, and the process starts all over again.

If the solution is explosive, it must eventually enter $\mathbb{R}^4_+$ or $\mathbb{R}^4_-$. In this case, the process cannot continue indefinitely; it terminates at some point, and S consists of a finite number of consecutive nodal cycles, followed by exactly one of the four possible "terminal sequences." That is, $\boldsymbol{v}$ satisfies Condition (d). If the solution is global, it cannot enter $\mathbb{R}^4_+$ or $\mathbb{R}^4_-$. In this case, the process must continue indefinitely, and S consists of an infinite number of consecutive nodal cycles. That is, $\boldsymbol{v}$ satisfies Condition (c).

As noted earlier, Condition (a) cannot arise if $\mu < \mu^*$, and Condition (c) is impossible if $\mu \geqslant \mu^*$. Now suppose that $\mu \geqslant \mu^*$ and that $\boldsymbol{v}$ satisfies Condition (d). Let $\bar{\boldsymbol{v}} = (\bar{v}_0, \bar{v}_1, \bar{v}_2, \bar{v}_3)$ be the global solution of (2.2) with $\bar{v}_0(0) = v_0(0)$. Then $\bar{\boldsymbol{v}}$ satisfies (a), necessarily with $\bar{v}_0 > 0$ and $\bar{v}_2 < 0$ throughout. If $v_2(0) > \bar{v}_2(0)$, then $\boldsymbol{v} > \bar{\boldsymbol{v}}$ on $(0, r_\infty)$, and since $\bar{v}_0$ is positive throughout, so is v_0. This eliminates all possible sequences of sign changes in (d), except for the sequence $\langle 2, 1\rangle$. If $v_2(0) < \bar{v}_2(0)$, then $\boldsymbol{v} < \bar{\boldsymbol{v}}$ on $(0, r_\infty)$, and since $\bar{v}_2$ is negative throughout, so is v_2. This eliminates all possible sequences of sign changes in (d), except for the sequence $\langle 0, 3\rangle$. The proof is thereby complete. □

Remark 5.5. (a) Let $\boldsymbol{v} = (v_0, v_1, v_2, v_3)$ be a nontrivial solution of (2.2), satisfying Condition (a) of Theorem 5.4. The proof shows that, up to symmetry, the nodal state of $\boldsymbol{v}$ is described by Diagrams 1 and 2; in particular, v_0 is positive and decreasing, v_2 negative and increasing, and v_1 and v_3 vanish only at 0. By Remark 2.7, it follows that $\boldsymbol{v}$ vanishes at infinity, with decay rates given by (2.4). Like the weaker assertion of Lemma 5.1, the corresponding characterization of the nontrivial global solutions of Eq. (2.1) in the case $\mu \geqslant \mu^*$ is well known (see [5,15] or Proposition 3.1 in [3]).

(b) Let $\boldsymbol{v}$ be a nontrivial solution of (2.2), satisfying Condition (b) of Theorem 5.4. Then $\boldsymbol{v}$ is either nonnegative throughout or nonpositive throughout. In either case, the strong comparison principle implies that the components of $\boldsymbol{v}$ never vanish, except possibly at 0.

We call a solution of (2.2) *sign-changing* if at least one of its components changes sign. The solution then satisfies one of Conditions (c) or (d) of Theorem 5.4, and the proof yields precise information about the nature of the zeros of the components. To facilitate the discussion, given a solution $\boldsymbol{v} = (v_0, v_1, v_2, v_3)$, "the component preceding v_0" is understood to be v_3.

Corollary 5.6. *Let $\boldsymbol{v}$ be a sign-changing solution of* (2.2)*.*

(a) *If $\boldsymbol{v}$ is a global solution, then all the zeros of all its components are simple.*
(b) *If $\boldsymbol{v}$ is explosive, then all the zeros of all its components are simple, with the possible exception of the point where the solution enters $\mathbb{R}^4_+$ or $\mathbb{R}^4_-$. This point is a simple zero of the last component to change sign and, possibly, an even zero of the preceding component, but not a zero of the other components.*

Proof. While carrying out the detailed analysis of the possible sequences of sign changes in the proof of Theorem 5.4, we explicitly noted that all the zeros of all the components of a sign-changing solution are necessarily simple, as long as the solution does not enter $\mathbb{R}^4_+$ or $\mathbb{R}^4_-$ (see Cases 1–3 and 1′–3′). This implies (a).

If the solution is explosive, it enters $\mathbb{R}^4_+$ or $\mathbb{R}^4_-$ at the point where the last sign change occurs. As we noted in the proof, this point is a simple zero of the component changing sign and may coincide with an even zero of the preceding component, but it is not a zero of the other two components (see Cases 2/3 and 2′/3′). This implies (b). □

The oscillation theorem for nontrivial global solutions of Eq. (2.1) in the subcritical case is now easy to prove.

Proof of Theorem 5.2. Assume that $\mu \in [0, \mu^*)$ and let u be a nontrivial global solution of (2.1). The corresponding solution of (2.2), whose components are u, $\partial_r u$, $(\partial_r^2 + \frac{\mu}{r}\partial_r)u$, and $\partial_r(\partial_r^2 + \frac{\mu}{r}\partial_r)u$, is global and therefore satisfies Condition (c) of Theorem 5.4; that is, the sequence of its sign changes consists of infinitely many consecutive nodal cycles. In each cycle, first u and $(\partial_r^2 + \frac{\mu}{r}\partial_r)u$ change sign, in either order or simultaneously, then $\partial_r u$ and $\partial_r(\partial_r^2 + \frac{\mu}{r}\partial_r)u$ change sign, in either order or simultaneously. In particular, both u and $\partial_r u$ have infinitely many zeros, all simple (by Corollary 5.6), and occurring in alternating order, with any two consecutive zeros of $\partial_r u$ separated by exactly one zero of u. Therefore, all critical points of u are either positive local maxima or negative local minima. □

The following result expresses a continuity or stability property of the number of sign changes of explosive solutions of (2.2) as a function of their initial values. For convenience, we call an explosive solution of (2.2) *critical* if it is sign-changing and enters $\mathbb{R}^4_+$ or $\mathbb{R}^4_-$ at an even zero of one of its components, or it is not sign-changing and one of the two even components vanishes at $r = 0$. Further, given a solution $\boldsymbol{v}$ of (2.2), by a *near-by solution* we mean a solution $\tilde{\boldsymbol{v}}$ with $\tilde{\boldsymbol{v}}(0)$ sufficiently close to $\boldsymbol{v}(0)$.

Corollary 5.7. *Let $\boldsymbol{v} = (v_0, v_1, v_2, v_3)$ be an explosive solution of* (2.2)*.*

(a) *Unless $\boldsymbol{v}$ is critical, every near-by solution has the same sequence of sign changes as $\boldsymbol{v}$.*
(b) *If $\boldsymbol{v}$ is critical, then every near-by solution has either the same sequence of sign changes as $\boldsymbol{v}$ or exactly two additional sign changes.*
(c) *Suppose that $\boldsymbol{v}$ is critical and sign-changing, with the last sign change occurring in the component v_i, for some $i \in \{0, 1, 2, 3\}$, so that the sequence of sign changes of $\boldsymbol{v}$ is of the form $\langle \dots, i\rangle$. Then the sequence of sign changes of any near-by solution is either $\langle \dots, i\rangle$ or $\langle \dots, i-1, i, i-1\rangle$, with the understanding that $i - 1 = 3$ if $i = 0$.*
(d) *Suppose $\boldsymbol{v}$ is critical and not sign-changing. If $v_0(0) = 0$ ($v_2(0) = 0$), then every near-by solution either has no sign changes, or it has exactly two sign changes, given by the sequence $\langle 0, 3\rangle$ (the sequence $\langle 2, 1\rangle$).*

Proof. Without loss of generality, we assume the solution $\boldsymbol{v} = (v_0, v_1, v_2, v_3)$ to be positive-explosive.

First suppose that $\boldsymbol{v}$ is sign-changing. Unless $\boldsymbol{v}$ enters $\mathbb{R}^4_+$ at an even zero of one of its components, Corollary 5.6 says that all the zeros of all the components are simple; beyond the last zero, the solution is positive. As a consequence of continuous dependence on initial values, the components of any near-by solution then have simple zeros close to those of the corresponding components of $\boldsymbol{v}$ and no additional zeros; in particular, every near-by solution has the same sequence of sign changes as $\boldsymbol{v}$. This proves (a) for the case of a sign-changing solution.

Now suppose that $\boldsymbol{v}$ is critical and that the last sign change of $\boldsymbol{v}$ occurs in the component v_i, for some $i \in \{0, 1, 2, 3\}$, necessarily at a point r_0 that is a simple zero of v_i and an even zero of v_{i-1} (v_3 in case that $i = 0$). By Corollary 5.6, all the preceding zeros of all the components of $\boldsymbol{v}$ are simple. With the same rationale as before, we infer that the components of any near-by solution $\tilde{\boldsymbol{v}} = (\tilde{v}_0, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$ have simple zeros close to those of the corresponding components of $\boldsymbol{v}$ and no additional zeros, except that the component $\tilde{v}_{i-1}$ may have an even zero or two simple zeros close to r_0. In the latter case, the two additional zeros of $\tilde{v}_{i-1}$ must be separated by a zero of $\tilde{v}_i$, necessarily the last one. (Note that this argument does not require r_0 to be a *double* zero of v_{i-1}; it is not if $i=0$. The important point is that r_0 is a *simple* zero of v_i.) It follows that the sequence of sign changes of $\tilde{\boldsymbol{v}}$ either coincides with that of $\boldsymbol{v}$ or is obtained from the latter by replacing the last term, i, with the sequence $\langle i-1, i, i-1\rangle$. This proves (c), and also (b), for the case of a sign-changing solution.

Next assume that $\boldsymbol{v}$ is not sign-changing, whence nonnegative throughout. If $v_0(0)$ and $v_2(0)$ are both positive, then every near-by solution starts and stays in $\mathbb{R}^4_+$ and hence has no sign changes. This proves (a) for the case of a solution that does not change sign. If $v_0(0)>0$ and $v_2(0)=0$, then a near-by solution $\tilde{\boldsymbol{v}}=(\tilde{v}_0,\tilde{v}_1,\tilde{v}_2,\tilde{v}_3)$ has no sign changes if $\tilde{v}_2(0)\geqslant 0$, but must have at least one sign change otherwise. However, no sign change occurs in $\tilde{v}_0$. Indeed, the component v_0 is increasing and hence bounded away from zero, which implies positivity of $\tilde{v}_0$ whenever $\tilde{\boldsymbol{v}}(0)$ is sufficiently close to $\boldsymbol{v}(0)$. This eliminates all but one of the possible sequences of sign changes in Theorem 5.4(d), and it follows that the sequence of sign changes of $\tilde{\boldsymbol{v}}$ is given by $\langle 2,1\rangle$. Similarly, if $v_2(0)>0$ and $v_0(0)=0$, then a near-by solution $\tilde{\boldsymbol{v}}=(\tilde{v}_0,\tilde{v}_1,\tilde{v}_2,\tilde{v}_3)$ has no sign changes if $\tilde{v}_0(0)\geqslant 0$, but exactly two sign changes, given by the sequence $\langle 0,3\rangle$, otherwise. This proves (d), and also (b), for the case of a solution that does not change sign. □

Remark 5.8. Let $\boldsymbol{v}$ be a critical positive-explosive solution of (2.2), $\tilde{\boldsymbol{v}}$ a near-by solution. In conjunction with the comparison principle, the proof of Corollary 5.7, Parts (c) and (d), shows that $\tilde{\boldsymbol{v}}$ has the same sequence of sign changes as $\boldsymbol{v}$ if $\tilde{\boldsymbol{v}}(0)\geqslant\boldsymbol{v}(0)$, but has exactly two additional sign changes if $\tilde{\boldsymbol{v}}(0)\leqslant\boldsymbol{v}(0)$ and $\tilde{\boldsymbol{v}}(0)\neq\boldsymbol{v}(0)$. An analogous statement obviously holds for critical negative-explosive solutions. In particular, the number of sign changes of positive-explosive (negative-explosive) solutions of (2.2) is nonincreasing and semicontinuous from above (nondecreasing and semicontinuous from below) as a function of their starting-points.

Combining the results of this section with those of the previous one, we can classify the explosive solutions of (2.2), starting on a completely ordered curve as in Theorem 4.6, in terms of their sequences of sign changes. For simplicity, we consider only the case of the vertical line $\{(1,\beta)\mid\beta\in\mathbb{R}\}$. This case is of particular relevance, as it yields a complete classification of all explosive solutions of Eq. (2.1) with center value 1. The following theorem describes the wide spectrum of possible behavior of explosive solutions in the subcritical case.

Theorem 5.9. *Suppose that $\mu\in[0,\mu^*)$. For $\beta\in\mathbb{R}$, denote by $\boldsymbol{v}^\beta$ the solution of* (2.2) *starting at* $(1,\beta)$*, by S^β the sequence of its sign changes. Let $\bar\beta$ be the* (*negative*) *real number such that $\boldsymbol{v}^{\bar\beta}$ is global. Let $c:=\langle 0/2, 1/3\rangle$ denote a nodal cycle. For $k\in\mathbb{Z}_+$, let c^k denote the sequence of sign changes consisting of exactly k consecutive nodal cycles.*

(a) *If $\beta\in[0,\infty)$, then $S^\beta=\emptyset$; in fact, $\boldsymbol{v}^\beta$ is nonnegative throughout.*
(b) *There exists a sequence $(\beta_k^+)_{k\in\mathbb{N}}$ of negative real numbers, decreasing with limit $\bar\beta$, such that the following holds for every $j\in\mathbb{Z}_+$ (with $\beta_0^+:=0$):*
  - *if $\beta\in[\beta^+_{4j+1},\beta^+_{4j})$, then $S^\beta=\langle c^{2j},2,1\rangle$;*
  - *if $\beta\in[\beta^+_{4j+2},\beta^+_{4j+1})$, then $S^\beta=\langle c^{2j},0/2,1,0\rangle$;*
  - *if $\beta\in[\beta^+_{4j+3},\beta^+_{4j+2})$, then $S^\beta=\langle c^{2j+1},0,3\rangle$;*
  - *if $\beta\in[\beta^+_{4(j+1)},\beta^+_{4j+3})$, then $S^\beta=\langle c^{2j+1},0/2,3,2\rangle$.*
(c) *There exists a sequence $(\beta_k^-)_{k\in\mathbb{N}}$ of negative real numbers, increasing with limit $\bar\beta$, such that the following holds for every $j\in\mathbb{Z}_+$ (with $\beta_0^-:=-\infty$):*
  - *if $\beta\in(\beta^-_{4j},\beta^-_{4j+1}]$, then $S^\beta=\langle c^{2j},0,3\rangle$;*
  - *if $\beta\in(\beta^-_{4j+1},\beta^-_{4j+2}]$, then $S^\beta=\langle c^{2j},0/2,3,2\rangle$;*
  - *if $\beta\in(\beta^-_{4j+2},\beta^-_{4j+3}]$, then $S^\beta=\langle c^{2j+1},2,1\rangle$;*
  - *if $\beta\in(\beta^-_{4j+3},\beta^-_{4(j+1)}]$, then $S^\beta=\langle c^{2j+1},0/2,1,0\rangle$.*

Proof. Part (a) is clear; indeed, if $\beta \in [0,\infty)$, then $\boldsymbol{v}^\beta(0) \geqslant 0$ and hence $\boldsymbol{v}^\beta \geqslant 0$ throughout. Starting at the point $(1,0)$, the solution $\boldsymbol{v}^0$ still has no sign changes, but by Corollary 5.7(d) and Remark 5.8, all near-by solutions $\boldsymbol{v}^\beta$ with $\beta < 0$ have exactly two sign changes, given by the sequence $\langle 2,1\rangle$.

By Theorem 5.4, the global solution $\boldsymbol{v}^{\bar\beta}$ is oscillatory, with infinitely many nodal cycles. Continuous dependence on initial values implies that, as β approaches $\bar\beta$ from above or from below, the number of nodal cycles of $\boldsymbol{v}^\beta$ approaches infinity. More precisely, in view of Corollary 5.7 and Remark 5.8, as β decreases from ∞ to $\bar\beta$ or increases from $-\infty$ to $\bar\beta$, the number of sign changes of $\boldsymbol{v}^\beta$ grows without bound and monotonically, increasing by 2 at each of a necessarily infinite sequence of critical values of β, that is, values such that the solution $\boldsymbol{v}^\beta$ is critical.

In particular, since $\boldsymbol{v}^\beta$ has exactly two sign changes for β just below 0, we find a sequence $(\beta_k^+)_{k\in\mathbb{N}}$ of real numbers with $0 > \beta_1^+ > \beta_2^+ > \cdots > \bar\beta$ such that, for every $k \in \mathbb{N}$, the solution $\boldsymbol{v}^\beta$ with $\beta = \beta_k^+$ has exactly $2k$ sign changes, while all near-by solutions $\boldsymbol{v}^\beta$ with $\beta < \beta_k^+$ have exactly $2(k+1)$ sign changes. Clearly, $\beta_k^+ \to \bar\beta$ as $k \to \infty$.

Further, given the sequence of sign changes of $\boldsymbol{v}^\beta = (v_0^\beta, v_1^\beta, v_2^\beta, v_3^\beta)$ for β just below 0, Corollary 5.7(c) uniquely determines how S^β changes at each of the critical values below. In fact, we have $S^\beta = \langle 2,1\rangle$ for β just below 0, and hence for all $\beta \in [\beta_1^+, 0)$. The solution $\boldsymbol{v}^\beta$ with $\beta = \beta_1^+$ is critical, necessarily with an even zero in the first component, v_0^β. By Corollary 5.7(c), we get $S^\beta = \langle 2,0,1,0\rangle$ for β just below β_1^+, and hence $S^\beta = \langle 0/2, 1, 0\rangle$ for all $\beta \in [\beta_2^+, \beta_1^+)$. (Note that Theorem 5.4 does not guarantee that the order of the first two sign changes remains the same.)

The solution $\boldsymbol{v}^\beta$ with $\beta = \beta_2^+$ is critical, necessarily with an even zero in the last component, v_3^β. By Corollary 5.7(c), $S^\beta = \langle 0/2, 1, 3, 0, 3\rangle$ for β just below β_2^+, hence $S^\beta = \langle 0/2, 1/3, 0, 3\rangle = \langle c, 0, 3\rangle$ for all $\beta \in [\beta_3^+, \beta_2^+)$. By the same token, $S^\beta = \langle c, 0, 2, 3, 2\rangle$ for β just below β_3^+, hence $S^\beta = \langle c, 0/2, 3, 2\rangle$ for all $\beta \in [\beta_4^+, \beta_3^+)$. Finally, $S^\beta = \langle c, 0/2, 3, 1, 2, 1\rangle$ for β just below β_4^+ and $S^\beta = \langle c^2, 2, 1\rangle$ for all $\beta \in [\beta_5^+, \beta_4^+)$. At this point, the pattern starts to repeat; in fact, the situation just below β_4^+ is the same as just below $\beta_0^+ := 0$, except that every component has changed sign twice. By induction, we obtain (b).

To prove (c), we first show that the component v_2^β is negative throughout if β is sufficiently close to $-\infty$. We argue along the same lines as in the proof of Theorem 4.6. For $\beta \in (-\infty, 0)$, define $\xi_\beta := (|\beta|^{-q/(q+2)}, -1)$. Clearly, $\xi_\beta \to (0,-1)$ as $\beta \to -\infty$. Note that the point $(1,\beta)$ is the $|\beta|^{1/(q+2)}$-rescaling of ξ_β; hence $\boldsymbol{v}^\beta$ is the $|\beta|^{1/(q+2)}$-rescaling of the solution starting at ξ_β. Now, the solution $\boldsymbol{v} = (v_0, v_1, v_2, v_3)$ starting at $(0,-1)$ has $v_2 \leqslant -1$ throughout, which implies that every near-by solution $\tilde{\boldsymbol{v}} = (\tilde v_0, \tilde v_1, \tilde v_2, \tilde v_3)$ has $\tilde v_2 < 0$ throughout. This holds, in particular, for the solution starting at ξ_β, if β is sufficiently close to $-\infty$, and then also for its rescalings, including $\boldsymbol{v}^\beta$.

Since v_0^β must change sign for all $\beta \in (-\infty, \bar\beta)$, but v_2^β does not change sign if β is sufficiently close to $-\infty$, Theorem 5.4 shows that $S^\beta = \langle 0, 3\rangle$ for all β sufficiently close to $-\infty$. With the same rationale as in the proof of (b), we find a sequence $(\beta_k^-)_{k\in\mathbb{N}}$ of real numbers with $\beta_1^- < \beta_2^- < \cdots < \bar\beta$ and $\beta_k^- \to \bar\beta$ as $k \to \infty$ such that, for every $k \in \mathbb{N}$, the solution $\boldsymbol{v}^\beta$ with $\beta = \beta_k^-$ has exactly $2k$ sign changes, while all near-by solutions $\boldsymbol{v}^\beta$ with $\beta > \beta_k^-$ have exactly $2(k+1)$ sign changes. Given the sequence of sign changes of $\boldsymbol{v}^\beta$ for β close to $-\infty$, Corollary 5.7(c) uniquely determines how S^β changes at each of the critical values above, and (c) follows with the same arguments as in the proof of (b). $\square$

Corollary 5.10. *Under the assumptions of Theorem* 5.9, *let u^β denote the first component of $\boldsymbol{v}^\beta$, for $\beta \in \mathbb{R}$. The number of sign changes of u^β grows without bound as $\beta \to \bar\beta$, successively attaining the values* $0, 2, 4, \ldots$ *as β decreases from ∞ to $\bar\beta$ and the values* $1, 3, 5, \ldots$ *as β increases from $-\infty$ to $\bar\beta$.*

Proof. If $\beta \in [0,\infty)$, then u^β is positive, increasing from 1 to ∞. Further, with $(\beta_k^\pm)_{k\in\mathbb{Z}_+}$ as in Theorem 5.9, and given $j \in \mathbb{Z}_+$, u^β changes sign exactly $2j$ times if $\beta \in [\beta_{4j+1}^+, \beta_{4j}^+)$, exactly $2j+2$ times if $\beta \in [\beta_{4(j+1)}^+, \beta_{4j+1}^+)$, while u^β changes sign exactly $2j+1$ times if $\beta \in (\beta_{4j}^-, \beta_{4j+3}^-]$, exactly $2j+3$ times if $\beta \in (\beta_{4j+3}^-, \beta_{4(j+1)}^-]$. The assertions follow readily. $\square$

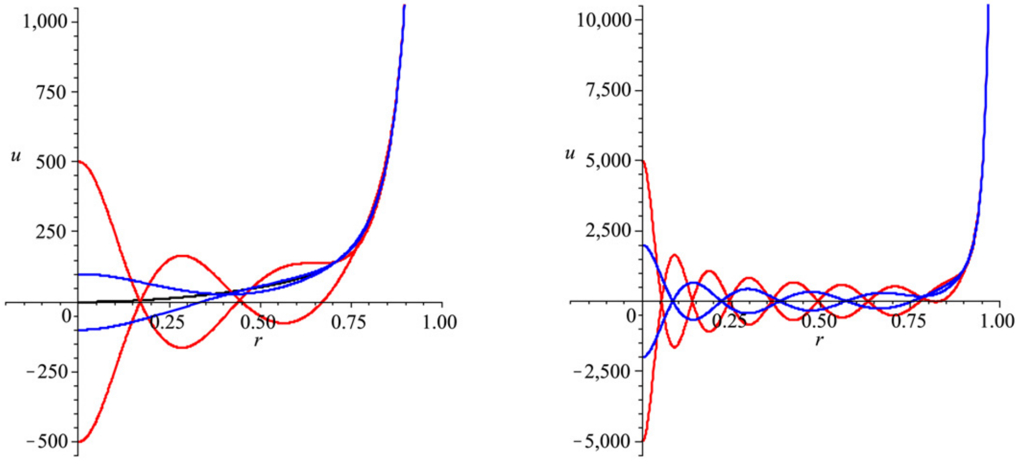


Fig. 3. Positive-explosive solutions of (2.1) with $p = 3$, $\mu = 2$ (subcritical).

Remark 5.11. Results analogous to Theorem 5.9 and Corollary 5.10 can also be stated in the critical/supercritical case. However, the range of possible behavior of explosive solutions is rather limited in this case, given the nature of the global solutions. Indeed, with the same notation as before, but assuming that $\mu \in [\mu^*, \infty)$, Theorem 5.4 yields the following classification. If $\beta \in [0, \infty)$, then $\boldsymbol{v}^\beta$ is nonnegative throughout, and u^β increases from 1 to ∞. If $\beta \in (\bar{\beta}, 0)$, then $S^\beta = \langle 2, 1\rangle$, and u^β decreases from 1 to a positive minimum, before increasing to ∞. If $\beta \in (-\infty, \bar{\beta})$, then $S^\beta = \langle 0, 3\rangle$, and u^β decreases from 1 to $-\infty$. Similar observations can be found in [5,8] and Section 4 of [3].

Remark 5.12. For $\alpha \in \mathbb{R}$ and $R \in (0, \infty)$, let $u^+_{\alpha,R}$ ($u^-_{\alpha,R}$) denote the unique positive-explosive (negative-explosive) solution of (2.1) with center value α and exit radius R (see Theorem 4.1). If $\alpha \in (0, \infty)$, then the $\alpha^{-1/q}$-rescaling of $u^+_{\alpha,R}$ ($u^-_{\alpha,R}$) is the first component of a solution of (2.2) starting at $(1, \beta)$ for some $\beta \in \mathbb{R}$ with $\beta > \bar{\beta}$ ($\beta < \bar{\beta}$). As α increases from 0 to ∞, so does the exit radius $R\alpha^{1/q}$ of the rescaling; hence its second center value β decreases from ∞ to $\bar{\beta}$ (increases from $-\infty$ to $\bar{\beta}$). In the subcritical case, it follows from Corollary 5.10 that the number of sign changes of $u^+_{\alpha,R}$ ($u^-_{\alpha,R}$) grows without bound as α increases from 0 to ∞, successively attaining the values $0, 2, 4, \ldots$ (the values $1, 3, 5, \ldots$). In the critical/supercritical case, Remark 5.11 implies that $u^+_{\alpha,R}$ ($u^-_{\alpha,R}$) is positive (changes sign exactly once) for every $\alpha \in (0, \infty)$. By symmetry, analogous statements hold for $\alpha \in (-\infty, 0)$. In any case, $u^+_{0,R}$ ($u^-_{0,R}$) does not change sign and vanishes only at 0. Figs. 3 and 4 show examples of the solutions $u^+_{\alpha,1}$ in a subcritical and a supercritical case, respectively.

Given a nontrivial solution u of Eq. (2.1) with exit radius r_∞, we say that u satisfies the Dirichlet condition at a point $r_0 \in (0, r_\infty)$ if r_0 is a common zero of u and $\partial_r u$. In view of Corollary 5.6, this is equivalent to saying that the corresponding solution $\boldsymbol{v} = (v_0, v_1, v_2, v_3)$ of (2.2) is sign-changing, explosive, and critical, with the last sign change occurring in v_1, at the point r_0. The following result is a consequence of this observation and Theorem 5.9.

Corollary 5.13. *Suppose that $\mu \in [0, \mu^*)$. For every $k \in \mathbb{Z}_+$ there exists a unique pair $\pm u_k$ of nontrivial solutions of* (2.1) *with exactly k sign changes in the interval* $(0, 1)$ *and satisfying the Dirichlet condition at $r = 1$. The center value $u_k(0)$ is positive and increases without bound as $k \to \infty$. Further, the rescaling of u_k with center value* 1 *converges, in the C^4-topology on compact subintervals of* $[0, \infty)$, *to the unique global solution of* (2.1) *with center value* 1.

Proof. Every nontrivial solution of (2.1) satisfying the Dirichlet condition at $r = 1$ has a positive or negative center value and corresponds to a sign-changing solution of (2.2) whose first component has

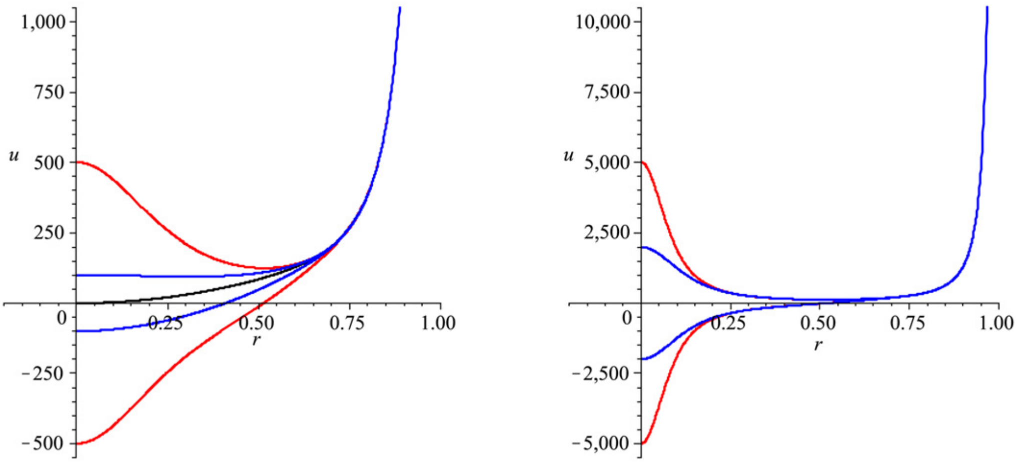


Fig. 4. Positive-explosive solutions of (2.1) with $p = 3$, $\mu = 12$ (supercritical).

an even zero at 1; up to symmetry, the latter is a rescaling of one of the solutions $\boldsymbol{v}^\beta$ of Theorem 5.9 with $\beta = \beta^+_{4j+1}$ or $\beta = \beta^-_{4j+3}$ for some $j \in \mathbb{Z}_+$. Note that, if $j \in \mathbb{Z}_+$ and $\beta = \beta^+_{4j+1}$, then $\boldsymbol{v}^\beta$ undergoes exactly $2j$ sign changes in the component v^β_0, before entering $\mathbb{R}^4_+$ at an even zero of v^β_0. If $\beta = \beta^-_{4j+3}$, then $\boldsymbol{v}^\beta$ undergoes exactly $2j+1$ sign changes in v^β_0, before entering $\mathbb{R}^4_-$ at an even zero of v^β_0.

Now, given $k \in \mathbb{Z}_+$, choose $j \in \mathbb{Z}_+$ such that $k = 2j$ or $k = 2j+1$ and let $\boldsymbol{v}_k := \boldsymbol{v}^\beta$ with $\beta := \beta^+_{4j+1}$ or $\beta := \beta^-_{4j+3}$, respectively. Then $\boldsymbol{v}_k$ undergoes exactly k sign changes in its first component, $v_{0,k}$, followed by an even zero of $v_{0,k}$. Since $\beta^+_{4j+1}, \beta^-_{4j+3} \to \bar\beta$ as $j \to \infty$, the solutions $\boldsymbol{v}_k$ converge, in the C^1-topology on compact subintervals of $[0,\infty)$, to the global solution $\boldsymbol{v}^{\bar\beta}$. For $k \in \mathbb{Z}_+$, let r_k denote the even zero of $v_{0,k}$. Then r_k increases without bound as $k \to \infty$. In fact, if $(\bar r_k)_{k\in\mathbb{Z}_+}$ is the increasing enumeration of the zeros of $v^{\bar\beta}_0$, then $r_k \in (\bar r_k, \bar r_{k+1})$ for every $k \in \mathbb{Z}_+$ (a consequence of the fact that the solutions $\boldsymbol{v}^\beta$ are completely ordered with respect to $\beta \in \mathbb{R}$). The assertions of the corollary now follow, with u_k the first component of the r_k-rescaling of $\boldsymbol{v}_k$. $\square$

The graph on the right of Fig. 5 shows the solutions u_0, u_1, u_2, u_3 of Corollary 5.13 in a special case. They are rescalings of four solutions with center value 1, which are depicted on the left, along with the unique global solution with center value 1.

We conjecture that an analogue of Corollary 5.13 holds under Navier rather than Dirichlet boundary conditions. A nontrivial solution u of (2.1) with exit radius r_∞ is said to satisfy the Navier condition at a point r_0 in $(0, r_\infty)$ if r_0 is a common zero of u and $(\partial^2_r + \frac{\mu}{r}\partial_r)u$, that is, a common zero of the components v_0 and v_2 of the corresponding solution $\boldsymbol{v} = (v_0, v_1, v_2, v_3)$ of the system (2.2).

In the situation of Theorem 5.9, the kth zero of v^β_0, for $k \in \mathbb{N}$, first appears for β just below β^+_{4j+1} if $k = 2j+1$, just above β^-_{4j+3} if $k = 2j+2$; at least for β close to the respective critical value, this zero trails the kth zero of v^β_2. Closer to $\bar\beta$, the two zeros may merge and possibly reverse order; if and where they merge, v^β_0 satisfies the Navier condition.

In general, we are unable to show that the relevant zeros do, in fact, merge. For the autonomous case, $\mu = 0$, we prove in [14] that the global solution $\boldsymbol{v}^{\bar\beta}$ is periodic, with simultaneous sign changes in the even components; thus $v^{\bar\beta}_0$ satisfies the Navier condition at each of its zeros. For $\mu > 0$, numerical evidence supports the conjecture that the kth zero of $v^{\bar\beta}_2$, for every $k \in \mathbb{N}$, trails that of $v^{\bar\beta}_0$; this would imply that the kth zeros of v^β_0 and v^β_2 must coincide for some β between $\bar\beta$ and β^+_{4j+1} if $k = 2j+1$, between $\bar\beta$ and β^-_{4j+3} if $k = 2j+2$. Further investigation is warranted.

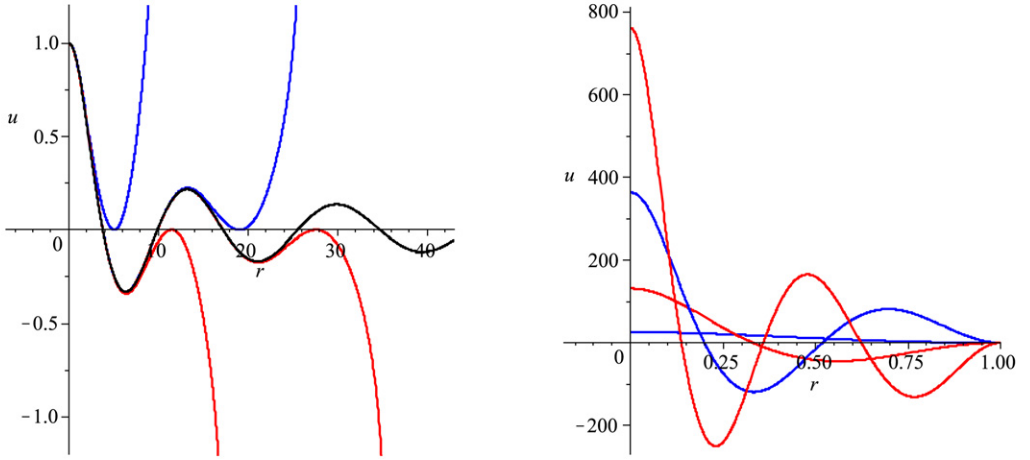


Fig. 5. Solutions of the Dirichlet problem for (2.1) with $p = 3$ and $\mu = 2$.

6. Conclusions

The results about radial solutions of the biharmonic equation (1.1), stated in the introduction, are obtained by combining the results pertaining to Eq. (2.1) in Sections 4 and 5. For the reader's convenience, we give here the appropriate references. Henceforth we assume that $\mu = n - 1$, for some $n \in \mathbb{N}$. Recall that the inequalities $p < p^*$ and $p \geqslant p^*$, characterizing subcritical and critical/supercritical growth of the nonlinearity in Eq. (1.1), are equivalent to the inequalities $\mu < \mu^*$ and $\mu \geqslant \mu^*$, respectively.

Proof of Theorem 1.1. The existence and uniqueness of a solution $\bar{u}$ of Problem (1.2) follows from Theorem 4.1(a), with $\alpha = 1$. For the qualitative properties of $\bar{u}$, see Remark 5.5(a) in the case $p \geqslant p^*$, Theorem 5.2 in the case $p < p^*$. $\square$

Proof of Theorem 1.4. Given $\alpha \in \mathbb{R}$, the existence and uniqueness of a solution u_α^+ of Problem (1.3) follows from Theorem 4.1(b), with $R = 1$. The convergence of the rescaling with center value 1 to the solution $\bar{u}$ of Problem (1.2) is guaranteed by Corollary 4.7. Finally, the assertions about the number of sign changes of u_α^+ follow from the observations in Remark 5.12. $\square$

Proof of Theorem 1.6. The assertions of the theorem follow directly from Corollary 5.13. $\square$

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